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MATHEMATICS

UNIT-1 PARTIAL DIFERENTIATION

1st order P.D :-

$z = f(x,y)$

*1. $\left.\dfrac{\partial z}{\partial y}\right|_{\text{keeping y constt.}} = f_y(x,y) = \lim\limits_{\Delta y \to 0} \dfrac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = q$

*2. $\left.\dfrac{\partial z}{\partial x}\right|_{\text{keeping x constt.}} = f_x(x,y) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = p$

2nd order P.D :-

*3. $\left.\dfrac{\partial^2 z}{\partial x^2}\right|_{\text{keeping y constt}} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}(p) = f_{xx} = r$

*4. $\dfrac{\partial^2 z}{\partial x \partial y} = \left.\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)\right|_{\text{Keeping y constt.}} = \dfrac{\partial}{\partial x}(q) = f_{xy} = s$

*5. $\dfrac{\partial^2 z}{\partial y \partial x} = \left.\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)\right|_{\text{Keeping x constt.}} = \dfrac{\partial}{\partial y}(p) = f_{yx} = s$

Both gives the same value through diff processes.

*6. $\dfrac{\partial^2 z}{\partial y^2} = \left.\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)\right|_{\text{Keeping x constt}} = \dfrac{\partial}{\partial y}(q) = f_{yy} = t$

NOTE :- If ft is of m variables then, Total no. of $\boxed{m^n}$, n^{th} order partial diff / derivatives exists.

Imp 2/3M ques.

Q1. If $v=(x^2+y^2+z^2)^{\frac{m}{2}}$, find $m\,(m\neq 0)$ st $\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}+\dfrac{\partial^2 v}{\partial z^2}$

A1.

$$\frac{\partial v}{\partial x}=\frac{m}{\cancel{2}}(x^2+y^2+z^2)^{\frac{m}{2}-1}\cdot(\cancel{2}x)$$

$$\frac{\partial v}{\partial y}=m(x^2+y^2+z^2)^{\frac{m}{2}-1}\cdot y$$

$$\frac{\partial v}{\partial z}=m(x^2+y^2+z^2)^{\frac{m}{2}-1}\cdot z$$

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(m(x^2+y^2+z^2)^{\frac{m}{2}-1}\cdot x\right)$$

$$=m\left[\left(\frac{m-2}{\cancel{2}}\right)(x^2+y^2+z^2)^{\frac{m}{2}-2}(\cancel{2}x^2)+(x^2+y^2+z^2)^{\frac{m}{2}-1}\right]$$

$$\frac{\partial^2 v}{\partial y^2}=m\left[\left(\frac{m-2}{\cancel{2}}\right)(x^2+y^2+z^2)^{\frac{m}{2}-2}(\cancel{2}y^2)+(x^2+y^2+z^2)^{\frac{m}{2}-1}\right]\text{ -(ii)}$$

$$\frac{\partial^2 v}{\partial z^2}=m\left[\left(\frac{m-2}{\cancel{2}}\right)(x^2+y^2+z^2)^{\frac{m}{2}-2}(\cancel{2}z^2)+(x^2+y^2+z^2)^{\frac{m}{2}-1}\right]\text{ -(iii)}$$

(i) + (ii) + (iii)

$$0=m\left[(m-2)(x^2+y^2+z^2)^{\frac{m}{2}-2}(x^2+y^2+z^2)+3(x^2+y^2+z^2)^{\frac{m}{2}-1}\right]$$

$$0=m\left[(m-2)(x^2+y^2+z^2)^{\frac{m}{2}-1}+3(x^2+y^2+z^2)^{\frac{m}{2}-1}\right]$$

$$0=m(x^2+y^2+z^2)^{\frac{m}{2}-1}\left[(m-2)+3\right]$$

$$0=\underbrace{m}_{\neq 0}(m+1)\underbrace{(x^2+y^2+z^2)}_{\neq 0}{}^{m/2-1}$$

$$m+1=0\Rightarrow\boxed{m=-1}$$

Q2. If $u = \log(x^3+y^3+z^3-3xyz)$, find $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$

A2. $\frac{\partial u}{\partial x} = \frac{1\cdot(3x^2-3yz)}{(x^3+y^3+z^3-3xyz)}$ —(i)

$\frac{\partial u}{\partial y} = \frac{(3y^2-3xz)}{(x^3+y^3+z^3-3xyz)}$ —(ii) $\qquad \frac{\partial u}{\partial z} = \frac{(3z^2-3xy)}{(x^3+y^3+z^3-3xyz)}$ —(iii)

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u\right]$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

(i) + (ii) + (iii)

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x^3+y^3+z^3-3xyz)} = \frac{3}{(x+y+z)}$$

$$\boxed{x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2}+\frac{(-3)}{(x+y+z)^2}+\frac{(-3)}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2}$$

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Homogeneous function :-

$z = f(x,y)$

$$z = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n$$

$$= x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \cdots + a_n\left(\frac{y}{x}\right)^n\right]$$

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$$\underbrace{z = x^n \phi\left(\frac{y}{x}\right)}_{\text{order of Hom. ft } z = n} \quad \text{or} \quad z = y^n \psi\left(\frac{x}{y}\right)$$

Euler's Theorem

If $z = f(x,y)$ is a homog ft of order n then

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz} \quad -\text{(I)}$$

Proof :-

$z = f(x,y)$ is Homog ft of order n

$\therefore z = x^n \phi\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\times\left(\frac{-y}{x^2}\right)$$

$$= nx^{n-1}\phi\left(\frac{y}{x}\right) - x^{n-2}y\,\phi'\left(\frac{y}{x}\right) \quad -\text{(i)} \times x$$

$$\frac{\partial z}{\partial y} = x^n \phi'\left(\frac{y}{x}\right)\times\left(\frac{1}{x}\right) = x^{n-1}\phi'\left(\frac{y}{x}\right) \quad -\text{(ii)} \times y$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n\phi\left(\frac{y}{x}\right) - \cancel{x^{n-1}y\,\phi'\left(\frac{y}{x}\right)} + \cancel{x^{n-1}y\,\phi'\left(\frac{y}{x}\right)}$$

$$= nx^n\phi\left(\frac{y}{x}\right) = nz$$

$\Rightarrow$ P.D of (I) wrt x :-

$$1.\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x}$$

$$\boxed{x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} = (n-1)\frac{\partial z}{\partial x}} \quad -\text{(iii)}$$

$\Rightarrow$ P.D of (I) wrt y:-

$$x\frac{\partial^2 z}{\partial y\partial x} + \frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial y^2} = n\frac{\partial z}{\partial y}$$

$$\boxed{x\frac{\partial^2 z}{\partial y\partial x} + y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y}} \quad \text{—(iv)}$$

multiply (iii) $\times x$ + (iv) $\times y$

$$x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x\partial y} + xy\frac{\partial^2 z}{\partial y\partial x} + y^2\frac{\partial^2 z}{\partial y^2} = x(n-1)\frac{\partial z}{\partial x} + y(n-1)\frac{\partial z}{\partial y}$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]$$

$$= (n-1)(n)z$$

$$\boxed{x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z} \quad \text{—(II)}$$

Q1. If $z = \dfrac{\sqrt{x}+\sqrt{y}}{x^2+y^2} = \dfrac{\sqrt{x}\boxed{1+\sqrt{\frac{y}{x}}}}{x^2\left(1+\left(\frac{y}{x}\right)^2\right)} = x^{-3/2}\left(\dfrac{1+\left(\frac{y}{x}\right)^{1/2}}{1+\left(\frac{y}{x}\right)^2}\right)$

Order of Homog ft is $-3/2$

Show:- Euler's thm:- $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \left(\dfrac{-3}{2}\right)z$

& $x^2\dfrac{\partial^2 z}{\partial x^2} + y^2\dfrac{\partial^2 z}{\partial y^2} + 2xy\dfrac{\partial^2 z}{\partial x\partial y} = n(n-1)z = \left(\dfrac{-3}{2}\right)\left(\dfrac{-3}{2}-1\right)z$

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* (i) $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\left(\frac{-3}{2}\right)z$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(\frac{x^{1/2}+y^{1/2}}{x^2+y^2}\right)=\frac{(x^2+y^2)(\frac{1}{2}x^{-1/2})-(2x)(x^{1/2}+y^{1/2})}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial y}=\frac{(x^2+y^2)(\frac{1}{2}y^{-1/2})-(2y)(x^{1/2}+y^{1/2})}{(x^2+y^2)^2}$$

$$x\frac{\partial z}{\partial x}=\frac{(x^2+y^2)(\frac{1}{2}x^{1/2})-(2x^2)(x^{1/2}+y^{1/2})}{(x^2+y^2)^2}$$

$$y\frac{\partial z}{\partial y}=\frac{(\frac{1}{2}y^{1/2})(x^2+y^2)-(2y^2)(x^{1/2}+y^{1/2})}{(x^2+y^2)^2}$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{(x^2+y^2)(\frac{1}{2}x^{1/2})-(2x^2)(x^{1/2}+y^{1/2})+(\frac{1}{2}y^{1/2})(x^2+y^2)-(2y^2)(x^{1/2}+y^{1/2})}{(x^2+y^2)^2}$$

$$=\frac{\frac{1}{2}(x^2+y^2)(x^{1/2}+y^{1/2})-2(x^{1/2}+y^{1/2})(x^2+y^2)}{(x^2+y^2)^2}$$

$$=\frac{-3}{2}\frac{(x^{1/2}+y^{1/2})}{(x^2+y^2)}=\frac{-3}{2}z$$

Hence proved.

$$x^2\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}+2xy\frac{\partial^2 z}{\partial x\partial y}=(n)(n-1)(z)$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{(x^2+y^2)^2\left[(x^2+y^2)(\frac{1}{2}x^{-1/2})-(2x)(x^{1/2}+y^{1/2})\right]'-((x^2+y^2)^2)'\left[(x^2+y^2)(\frac{1}{2}x^{-1/2})-(2x)(x^{1/2}+y^{1/2})\right]}{(x^2+y^2)^4}$$

$$=\frac{(x^2+y^2)^2\left[(2x)(\frac{1}{2}x^{-1/2})+(x^2+y^2)(-\frac{1}{4}x^{-3/2})-2(x^{1/2}+y^{1/2})-2x(\frac{1}{2}x^{-1/2})\right]-2(x^2+y^2)(2x)\left[(x^2+y^2)(\frac{1}{2}x^{-1/2})-(2x)(x^{1/2}+y^{1/2})\right]}{(x^2+y^2)^4}$$

$$=\frac{(x^2+y^2)^2\left[x^{1/2}+(-\frac{1}{4})x^{1/2}-(\frac{1}{4})x^{-3/2}y^2-2x^{1/2}-2y^{1/2}-x^{1/2}\right]-4(x^3+xy^2)\left[\frac{1}{2}x^{3/2}+\frac{1}{2}x^{-1/2}y^2-2x^{3/2}-2xy^{1/2}\right]}{(x^2+y^2)^4}$$

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$u = \sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]$

u is not known as homog ft but it is a function of homogeneous expression within it.
u is called a ft of homog expression of DEGREE = 1 (N)

Take z = sin u = $\frac{x^2+y^2}{x+y} = f(u)$

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$

Take z = f(u) then $\frac{\partial z}{\partial u} = f'(u)$

then $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}$$

$$x f'(u)\frac{\partial u}{\partial z} + y f'(u)\frac{\partial u}{\partial y} = n f(u)$$

$$\boxed{x\frac{\partial u}{\partial z} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}}$$

n = Degree

Putting $g(u) = \frac{n f(u)}{f'(u)}$ in (II)

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u)-1\right]}$$

Eg:- for $u = \sin^{-1}\left[\frac{x^4+y^4}{x+y}\right]$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\frac{\sin u}{\cos u} = 3\tan u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 3\tan u\left[3\sec^2 u - 1\right]$$

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$$3\tan u\left[3\sec^2 u-1\right] = \left(3\frac{\sin u}{\cos u}\right)\left(\frac{3}{\cos^2 u}-1\right)$$

$$= \frac{9\sin u}{\cos^3 u} - \frac{3\sin u}{\cos u} = \frac{9\sin u - 3\sin u\cos^2 u}{\cos^3 u}$$

Q2. If $u = \log_e\left[\frac{x^4+y^4}{x+y}\right]$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f(u)'}$$

$$f(u) = \log_e u = \frac{x^4+y^4}{x+y}$$

$$= \frac{x^4\left(1+\left(\frac{y}{x}\right)^4\right)}{x\left(1+\left(\frac{y}{x}\right)\right)}$$

$$= x^3\frac{\left(1+\left(\frac{y}{x}\right)^4\right)}{\left(1+\left(\frac{y}{x}\right)\right)}$$

$$\boxed{\text{Degree} = 3}$$

Let $\log_e\left[\frac{x^4+y^4}{x+y}\right] = u$

$$\frac{x^4+y^4}{x+y} = e^u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\frac{e^u}{e^u} = 3$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$$

To prove the thrm; for Q1

$$u = \sin^{-1}\left[\frac{x^4+y^4}{x+y}\right] \Rightarrow \sin u = z = \frac{x^4+y^4}{x+y}$$

$$x\frac{\partial z}{\partial u}\times\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\times\frac{\partial u}{\partial y} = x(\cos u)\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial z}$$

$$= n\cancel{z}u$$

Ques Practice

Q. If $u = e^{xyz}$, find $\frac{\partial^3 u}{\partial x \partial y \partial z}$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)\right)$$

$$\frac{\partial u}{\partial z} = xy e^{xyz}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left(xy e^{xyz}\right)$$

$$\frac{\partial^2 u}{\partial y \partial z} = xe^{xyz} + x^2yze^{xyz} = xe^{xyz}(1+xyz)$$
$$= e^{xyz}(x+x^2yz)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(e^{xyz}(x+x^2yz)\right)$$
$$= yze^{xyz}(x+x^2yz) + e^{xyz}(1+2xyz)$$
$$= e^{xyz}(xyz+x^2y^2z^2+1+2xyz)$$
$$= e^{xyz}(x^2y^2z^2+3xyz+1)$$

Q2 If $x^x y^y z^z = c$, show that at $x=y=z$

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$$

A2. $\log x^x y^y z^z = \log c$

$\log x^x + \log y^y + \log z^z = \log c$

$x \log x + y \log y + z \log z = \log c$

Diff wrt x :-

$$\left(\log x + \frac{1}{x}\cdot x\right) + \left(\log z + z\frac{1}{z}\right)\frac{\partial z}{\partial x} = 0$$

$$(1+\log x) + (1+\log z)\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{(1+\log z)} \quad -(i)$$

$||ly$ $$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{(1+\log z)}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{-\partial}{\partial x}\left(\frac{1+\log y}{1+\log z}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -(1+\log y)\left[\frac{-1}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right] -(ii)$$

Putting (i) in (ii)

$$\frac{\partial^2 z}{\partial x \partial y} = +-\frac{(1+\log y)}{(1+\log z)^2}\left(\frac{1+\log x}{1+\log z}\right)\cdot\frac{1}{z}$$

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)_{at\ x=y=z} = \frac{-(1+\log x)^2}{x.(1+\log x)^3} = \frac{-1}{(\log_e e + \log_e x).x}$$

$$= \frac{-1}{x(\log ex)} = -(x\log ex)^{-1}$$

Q. If $z(x+y) = x^2+y^2$, Show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1-\frac{\partial z}{\partial x} - \dots\right.$

Ans. $z = \frac{x^2+y^2}{x+y}$

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$$\frac{\partial z}{\partial x} = \frac{(x+y)(2x) - (x^2+y^2)}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2} \quad -(i)$$

$$\text{IIly } \frac{\partial z}{\partial y} = \frac{y^2+2yx-x^2}{(x+y)^2} \quad -(ii)$$

LHS:- $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left(\frac{2x^2-2y^2}{(x+y)^2}\right)^2 = \frac{4(x^2-y^2)^2}{(x+y)^4}$$

$$= \frac{4(x-y)^2(x+y)^{\cancel{2}}}{(x+y)^4} = \frac{4(x-y)^2}{(x+y)^2}$$

RHS. $$4\left(1-\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left(1-\left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)\right)$$

$$= 4\left(1-\frac{4xy}{(x+y)^2}\right) = 4\left(\frac{(x+y)^2-4xy}{(x+y)^2}\right)$$

$$= 4\frac{(x-y)^2}{(x+y)^2}$$

$$\boxed{\text{LHS} = \text{RHS}}$$

Q4 $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, Prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial u}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 2y \cdot \tan^{-1}\left(\frac{x}{y}\right) - y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= \frac{x}{\cancel{x^3+y^3}}\left(\cancel{x^3+y^3}\right) - 2y\tan^{-1}\frac{x}{y}$$

$$= x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(x - 2y\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$= 1 - 2y \cdot \frac{1}{1+\frac{x^2}{y^2}} \times \frac{1}{y}$$

$\frac{4}{4} - \frac{1}{4} - \frac{8}{4}$

$$= 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

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$$\frac{\partial^2 z}{\partial x^2} = \frac{(x^2+y^2)^2\left(x^{1/2} - \frac{1}{4}x^{1/2} - \frac{1}{4}x^{-3/2}y^2 - 2x^{1/2} - 2y^{1/2} - x^{1/2}\right) - 4(x^3+xy^2)\left(\frac{1}{2}x^{3/2} + \frac{1}{2}x^{-1/2}y^2 - 2x^{3/2} - 2xy^{1/2}\right)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^2\left(-\frac{9}{4}x^{1/2} - \frac{1}{4}x^{-3/2}y^2 - 2y^{1/2}\right) - 4(x^3+xy^2)\left(-\frac{3}{2}x^{3/2} + \frac{1}{2}x^{-1/2}y^2 - 2xy^{1/2}\right)}{(x^2+y^2)^4}$$

$\neq \cancel{(x^2+y^2)[(x^2+y}$

$$= \frac{(x^2+y^2)\left(-\frac{9}{4}x^{1/2} - \frac{1}{4}x^{-3/2}y^2 - 2y^{1/2}\right) - 4(x)\left(-\frac{3}{2}x^{3/2} + \frac{1}{2}x^{-1/2}y^2 - 2xy^{1/2}\right)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(y^2+x^2)\left(-\frac{9}{4}y^{1/2} - \frac{1}{4}y^{-3/2}x^2 - 2x^{1/2}\right) - 4(y)\left(-\frac{3}{2}y^{3/2} + \frac{1}{2}y^{-1/2}x^2 - 2yx^{1/2}\right)}{(y^2+x^2)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{(x^2+y^2)(1/2\, y^{-1/2}) - (2y)(x^{1/2}+y^{1/2})}{(x^2+y^2)^2}\right)$$

$$= \frac{\left[(\cancel{2}x)(\cancel{1/2}\, y^{-1/2}) - (\cancel{2}y)(\cancel{1/2}\, x^{-1/2})\right](x^2+y^2)^2 - (2(x^2+y^2)(2x))\left[(x^2+y^2)(1/2\, y^{-1/2}) - (2y)(x^{1/2}+y^{1/2})\right]}{(x^2+y^2)^4}$$

$$= \frac{(xy^{-1/2} - yx^{-1/2})(x^2+y^2)^2 - 4x(x^2+y^2)\left[(x^2+y^2)(1/2\, y^{-1/2}) - (2y)(x^{1/2}+y^{1/2})\right]}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(xy^{-1/2} - yx^{-1/2}) - 4x(1/2\, x^2 y^{-1/2} + 1/2\, y^{3/2} - 2yx^{1/2} - 2y^{3/2}}{(x^2+y^2)^3}$$

$$= \frac{(x^2+y^2)(xy^{-1/2} - yx^{-1/2}) - 4x(1/2\, x^2 y^{-1/2} - 2yx^{1/2} - 3/2\, y^{3/2})}{(x^2+y^2)^3}$$

$$x^2\frac{\partial^2 z}{\partial x^2} = \frac{(x^2+y^2)(-9/4\, x^{5/2} - 1/4\, x^{1/2} - 2y^{1/2}x^2) - 4x^3(-3/2\, x^{3/2} + 1/2\, x^{-1/2}y^2 - 2xy^{1/2})}{(x^2+y^2)^3}$$

$$y^2\frac{\partial^2 z}{\partial x^2} = \frac{(y^2+x^2)(-9/4\, y^{5/2} - 1/4\, y^{1/2} - 2x^{1/2}y^2) - 4y^3(-3/2\, y^{3/2} + 1/2\, y^{-1/2}x^2 - 2yx^{1/2})}{(x^2+y^2)^3}$$

$$2xy\frac{\partial^2 z}{\partial x \partial y} = \frac{(x^2+y^2)(2x^2y^{1/2} - 2y^2x^{1/2}) - 8x(1/2\, x^3y^{1/2} - 2y^2x^{3/2} - 3/2\, xy^{5/2})}{(x^2+y^2)^3}$$

$$x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)(z) = +\frac{15}{4}(z)$$

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(6 marks)

Q1. If $u = \sin^{-1}[(x^2+y^2)^{1/5}]$ find $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$

$$z = (x^2+y^2)^{1/5} = \left[x^2\left(1+\frac{y^2}{x^2}\right)\right]^{1/5} = x^{2/5}\left[\left(1+\frac{y^2}{x^2}\right)^{1/5}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = g(u)\left[g'(u) - 1\right] \text{ — (i)}$$

where $g(u) = n\dfrac{f(u)}{f'(u)}$

$$g(u) = 2/5\ \frac{(x^2+y^2)^{1/5}}{1/5(x^2+y^2)^{-4/5}(2x)}$$

$$= \cancel{2}\frac{(x^2+y^2)^{1/5}(x^2+y^2)^{4/5}}{(\cancel{2}x)} = \frac{(x^2+y^2)}{(x)}$$

$$g'(u) = \frac{(2x)(x) - (x^2+y^2)}{(x)^2}$$

Putting in (i)

$$\frac{(x^2+y^2)}{(x)}\left[\left(\frac{2}{5}\right)\frac{2x^2 - x^2 - y^2}{x^2} - 1\right] \text{ Ans.}$$

$$\cancel{\frac{x^2+y^2}{x}\left[\frac{-y^2}{x^2}\right]}$$

$$z = \sin u = f(u) = \left[(x^2+y^2)^{1/5}\right]$$

$$g(u) = n\frac{(\sin u)}{\cos u} \qquad g'(u) = n\left[\frac{\cancel{\sin u(\sin u)}\cos u(\cos u) + \sin u^2}{(\cos u)^2}\right]$$

$$\therefore\ g(u)\left[g'(u) - 1\right] = n\left(\frac{\sin u}{\cos u}\right)\left[\frac{n}{(\cos u)^2} - 1\right]$$

$$= \frac{2}{5}\tan u\left(\frac{2}{5}\sec^2 u - 1\right)$$

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⇒ JACOBIANS:- (use in variable transformation)

$z = f(x, y)$

$u = \phi(x, y)$ & $v = \psi(x, y)$

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \partial = \frac{\partial(u, v)}{\partial(x, y)} = \partial\left(\frac{u, v}{x, y}\right)$$

Q1 $x = r\cos\theta$ $y = r\sin\theta$

find Jacobian $\left(\frac{x, y}{r, \theta}\right)$ & $J\left(\frac{r, \theta}{x, y}\right)$

Ans. $\frac{\partial x}{\partial r} = \cos\theta$; $\frac{\partial x}{\partial \theta} = -r\sin\theta$; $\frac{\partial y}{\partial r} = \sin\theta$; $\frac{\partial y}{\partial \theta} = r\cos\theta$;

$$J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

$$J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix}$$

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

$$\boxed{r = \sqrt{x^2 + y^2}}$$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}}(x^2 + y^2)^{-1/2}(\cancel{2}x) \qquad \frac{\partial r}{\partial y} = \frac{1}{\cancel{2}}(x^2 + y^2)^{-1/2}(\cancel{2}y)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + y^2/x^2}\left(\frac{-y}{x^2}\right) \qquad \frac{\partial \theta}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{1}{x}\right)$$

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$$\frac{\partial r}{\partial x} = x(x^2+y^2)^{-1/2} \qquad \frac{\partial r}{\partial y} = y(x^2+y^2)^{-1/2}$$

$$\frac{\partial \theta}{\partial x} = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) \qquad \frac{\partial \theta}{\partial xy} = \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$J\left(\frac{r,\theta}{x,y}\right) = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$\frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{(x^2+y^2)}} = \frac{1}{r}$$

Properties of Jacobians :-

(1.) If $u = \phi(x,y)$ & $v = \psi(x,y)$ [Prove at home]

then $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

(2.) If $u = f(x,y)$ & $v = g(x,y)$ where $x = \phi(s,t)$ & $y = \psi(s,t)$

Then $\boxed{\frac{\partial(u,v)}{\partial(s,t)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(s,t)}}$

(3.) If u, v, w are ft of x, y, z s.t. u, v, w are functionally dependent then,

$$\boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0}$$

Q. If $y_1 = \dfrac{x_2 x_3}{x_1}$; $y_2 = \dfrac{x_1 x_3}{x_2}$; $y_3 = \dfrac{x_1 x_2}{x_3}$

find $J\left(\dfrac{x_1, x_2, x_3}{y_1, y_2, y_3}\right)$

Ans.
$$J\left(\frac{y_1, y_2, y_3}{x_1, x_2, x_3}\right) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$\frac{\partial y_1}{\partial x_1} = \frac{-(x_2 x_3)}{(x_1)^2} \qquad \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1} \qquad \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2} \qquad \frac{\partial y_2}{\partial x_2} = \frac{-(x_1 x_3)}{(x_2)^2} \qquad \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3} \qquad \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3} \qquad \frac{\partial y_3}{\partial x_3} = \frac{-(x_1 x_2)}{(x_3^2)}$$

$$J\left(\frac{y_1, y_2, y_3}{x_1, x_2, x_3}\right) = \begin{vmatrix} -(x_2 x_3)/(x_1)^2 & x_3/x_1 & x_2/x_1 \\ x_3/x_2 & -(x_1 x_3)/(x_2)^2 & x_1/x_2 \\ x_2/x_3 & x_1/x_3 & -(x_1 x_2)/(x_3)^2 \end{vmatrix}$$

$$= -\frac{(x_2 x_3)}{(x_1)^2}\left[\left(\frac{x_1 x_3}{x_2^2}\right)\left(\frac{x_1 x_2}{(x_3)^2}\right) - \left(\frac{x_1}{x_2}\right)\left(\frac{x_1}{x_3}\right)\right]$$
$$- \frac{x_3}{x_1}\left[-\left(\frac{x_2 x_3}{(x_2)^2}\right)\left(\frac{x_1 x_2}{(x_3)^2}\right) - \left(\frac{x_1}{x_2}\right)\left(\frac{x_2}{x_3}\right)\right]$$
$$+ \frac{x_2}{x_1}\left[\left(\frac{x_3}{x_2}\right)\left(\frac{x_1}{x_3}\right) + \left(\frac{x_1 x_3}{(x_2)^2}\right)\left(\frac{x_2}{x_3}\right)\right] = 2+1+1 = 4$$

$$J\left(\frac{x_1, x_2, x_3}{y_1, y_2, y_3}\right) = \frac{1}{J\left(\frac{y_1, y_2, y_3}{x_1, x_2, x_3}\right)} = \frac{1}{4}$$

PROOF :-

(a) $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

Let $u = f(x,y)$ $v = \phi(x,y)$

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

ALL DIFFERENTIATION FORMULA:-

(a) $\frac{d}{dx}(x) = 1$

(b) $\frac{d}{dx}(ax) = a$

(c) $\frac{d}{dx}(x^n) = nx^{n-1}$

(d) $\frac{d}{dx}(\cos x) = -\sin x$

(e) $\frac{d}{dx}(\sin x) = \cos x$

(f) $\frac{d}{dx}(\tan x) = \sec^2 x$

(g) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(h) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(i) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(j) $\frac{d}{dx}(\ln x) = \frac{1}{x}$

(k) $\frac{d}{dx}(e^x) = e^x$

(l) $\frac{d}{dx}(a^x) = a^x \ln a$

(m) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

(n) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

(o) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$

(p) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

Ques practice :-

(Doubt) Q. $u^3+v^3 = x+y$, $u^2+v^2 = x^3+y^3$

Show $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1(y^2-x^2)}{2uv(u-v)}$

$f_1 = u^3+v^3-x-y=0$

$f_2 = u^2+v^2-x^3-y^3=0$

$$J = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

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Q. If $u = xyz \quad v = x^2+y^2+z^2 \quad w = x+y+z$

Find Jacobian of $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = yz \qquad \frac{\partial u}{\partial y} = xz \qquad \frac{\partial u}{\partial z} = xy$$

$$\frac{\partial v}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2y \qquad \frac{\partial v}{\partial z} = 2z$$

$$\frac{\partial w}{\partial x} = 1 \qquad \frac{\partial w}{\partial y} = 1 \qquad \frac{\partial w}{\partial z} = 1$$

$$= yz(2y-2z) - xz(2x-2z) + xy(2x-2y)$$

$$= 2[yz(y-z) - xz(x-z) + xy(x-y)]$$

$$= -2(x-y)(y-z)(z-x)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-1}{2(x-y)(y-z)(z-x)}$$

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Q2. If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$ & $y = r\sin\theta$
Find $\frac{\partial(u,v)}{\partial(r,\theta)}$

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(r,\theta)}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = 2x$; $\frac{\partial u}{\partial y} = -2y$; $\frac{\partial v}{\partial x} = 2y$; $\frac{\partial v}{\partial y} = 2x$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2 + 4y^2 = 4(x^2+y^2)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$\frac{\partial x}{\partial r} = \cos\theta$; $\frac{\partial x}{\partial \theta} = -r\sin\theta$; $\frac{\partial y}{\partial r} = \sin\theta$; $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = 4r(x^2+y^2)$$

$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) = 4r^2$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3 (\cancel{x^2} +$$

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Q3 If $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$, $w = x + y + z$
Determine whether there exists a functional relationship b/w u, v & w and if so, find it.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = y+z \qquad \frac{\partial u}{\partial y} = x+z \qquad \frac{\partial u}{\partial z} = y+x$

$\frac{\partial v}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2y \qquad \frac{\partial v}{\partial z} = 2z$

$\frac{\partial w}{\partial x} = 1 \qquad \frac{\partial w}{\partial y} = 1 \qquad \frac{\partial w}{\partial z} = 1$

$$= \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= y+z(2y-2z) - (x+z)(2x-2z) + (y+x)(2x-2y)$$

$$= 2[y^2 - z^2 - (x^2 - z^2) - y^2 + x^2]$$

$$= 0$$

There exists a reln b/w them;

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$$

$$w^2 = v + 2u.$$

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Q4. $u = \dfrac{x+y}{1-xy}$ $\qquad v = \tan^{-1}x + \tan^{-1}y$ $\qquad \boxed{v = \tan^{-1}u}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)-(-y)(x+y)}{(1-xy)^2} = \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)-(-x)(x+y)}{(1-xy)^2} = \frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} \qquad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

relation $\Rightarrow v = \tan^{-1}u$

Date :- 26 NOV, 2022

⇒ Jacobians of Implicit functions.

$f_1(x,y,u,v) = 0$ -(i) where $u = u(x,y)$ & $v = v(x,y)$

$f_2(x,y,u,v) = 0$ -(ii)

Partially diff (i) & (ii) wrt x & y both :-

Equation -(i)

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \quad \text{— (I)}$$

$$\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \quad \text{— (II)}$$

Equation -(ii)

$$\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \quad \text{— (III)}$$

$$\frac{\partial f_2}{\partial y} + \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \quad \text{— (IV)}$$

To find : ~~$\partial(x,y)$~~ $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(f_1,f_2)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)}$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} \times \frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}$$

$$= \left(\frac{\partial f_1}{\partial u} \times \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial v} * \frac{\partial f_2}{\partial x}\right) + \left(\frac{\partial f_1}{\partial u}\cdot\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial f_2}{\partial y}\right)$$
$$+ \left(\frac{\partial f_2}{\partial u}\cdot\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial f_2}{\partial x}\right) + \left(\frac{\partial f_2}{\partial u}\cdot\frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial f_2}{\partial y}\right)$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u}\cdot\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial f_2}{\partial x} & \frac{\partial f_1}{\partial u}\cdot\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial f_2}{\partial y} \\ \frac{\partial f_2}{\partial u}\cdot\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial u}\cdot\frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial f_2}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x} & \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial y} \\ \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x} & \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial y} \end{vmatrix}$$

using (I), (II), (III), (IV)

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$$= \begin{vmatrix} -\frac{\partial f_1}{\partial x} & -\frac{\partial f_1}{\partial y} \\ -\frac{\partial f_2}{\partial x} & -\frac{\partial f_2}{\partial y} \end{vmatrix} = (-1)^2 \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}$$

$$= (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x, y)}$$

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f_1, f_2)/\partial(x,y)}{\partial(f_1,f_2)/\partial(u,v)}}$$

If $f_1, f_2, \dots, f_n$ are implicit functions of $x_1, x_2, \dots x_n, u_1, u_2, \dots, u_n$, then

$$\boxed{\frac{\partial(u_1, u_2, \cdots u_n)}{\partial(x_1, x_2, \dots, x_n)} = (-1)^n \frac{\dfrac{\partial(f_1, f_2, \cdots f_n)}{\partial(x_1, x_2, \dots, x_n)}}{\dfrac{\partial(f_1, f_2, \dots, f_n)}{\partial(u_1, u_2, \dots, u_n)}}}$$

⇒ Partial differentiation of Implicit functions

To find :- $\frac{\partial u}{\partial x}$; $\frac{\partial v}{\partial x}$ ---

To find :- $X = \frac{\partial u}{\partial x}$; $Y = \frac{\partial v}{\partial x}$

using equation (I) & (III),

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$$

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$$\frac{\frac{\partial u}{\partial x}}{-\left(\frac{\partial f_1}{\partial x}\cdot\frac{\partial f_2}{\partial v}-\frac{\partial f_1}{\partial v}\cdot\frac{\partial f_2}{\partial x}\right)} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial f_1}{\partial x}\cdot\frac{\partial f_2}{\partial u}-\frac{\partial f_1}{\partial u}\cdot\frac{\partial f_2}{\partial x}}$$

$$= \frac{1}{\frac{\partial f_1}{\partial u}\cdot\frac{\partial f_2}{\partial v}-\frac{\partial f_1}{\partial v}\cdot\frac{\partial f_2}{\partial u}}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{-\frac{\partial(f_1,f_2)}{\partial(x,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}} \qquad \boxed{\frac{\partial v}{\partial x} = \frac{-\frac{\partial(f_1,f_2)}{\partial(u,x)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}}$$

$$\boxed{\frac{\partial x}{\partial u} = \frac{-\frac{\partial(f_1,f_2)}{\partial(u,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}} \qquad \boxed{\frac{\partial y}{\partial v} = \frac{-\frac{\partial(f_1,f_2)}{\partial(v,x)}}{\frac{\partial(f_1,f_2)}{\partial(x,y)}}}$$

Q. find $\frac{\partial u}{\partial x}$:- $u^2+xv^2-xy=0$

and $u^2+xyv+v^2=0$

$f_1 = u^2+xv^2-xy=0 \qquad f_2 = u^2+xyv+v^2=0$

$$\frac{\partial u}{\partial x} = \frac{-\frac{\partial(f_1,f_2)}{\partial(x,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}} \quad — (c)$$

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$$\frac{\partial(f_1,f_2)}{\partial(x,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v^2-y & 2xv \\ yv & xy+2v \end{vmatrix}$$

$$= (v^2-y)(xy+2v) - 2xv(yv) \rightarrow (A)$$

$\frac{\partial f_1}{\partial x} = v^2 - y$ $\qquad$ $\frac{\partial f_2}{\partial x} = yv$

$\frac{\partial f_1}{\partial v} = 2xv$ $\qquad$ $\frac{\partial f_2}{\partial v} = xy + 2v$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \cancel{v^2-y}$$

$$= \begin{vmatrix} 2u & 2xv \\ 2u & xy+2v \end{vmatrix}$$

$$= 2u(xy+2v) - 2xv(2u) \rightarrow (B)$$

$\frac{\partial f_1}{\partial u} = 2u$ $\qquad$ $\frac{\partial f_2}{\partial u} = 2u$

$\frac{\partial f_1}{\partial v} = 2xv$ $\qquad$ $\frac{\partial f_2}{\partial v} = xy + 2v$

Solving (A)

$$xyv^2 - xy^2 + 2v^3 - 2yv - 2xyv^2$$

Solving (B)

$$2uxy + 4uv - 4xvu$$

Putting in (C)

$$\frac{\partial u}{\partial x} = -\frac{(xyv^2 - xy^2 + 2v^3 - 2yv - 2xyv^2)}{(2uxy + 4uv - 4xvu)}$$

$$= \frac{-xyv^2 + xy^2 - 2v^3 + 2yv + 2xyv^2}{2uxy + 4uv - 4xvu} \quad \text{Ans.}$$

Q1 If $u = x + y^2$ find $\dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u}$

$v = y + z^2$

$w = z + x^2$

Ans $f_1(x, y, z, u, v, w) = u - x - y^2 = 0$

$f_2(x, y, z, u, v, w) = v - y - z^2 = 0$

$f_3(x, y, z, u, v, w) = w - z - x^2 = 0$

$$\frac{\partial x}{\partial u} = -\frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}} = -$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} \div \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2y & 0 \\ 0 & -1 & -2z \\ 0 & 0 & -1 \end{vmatrix} \div \begin{vmatrix} -1 & -2y & 0 \\ 0 & -1 & -2z \\ -2x & 0 & -1 \end{vmatrix}$$

$$= 1(-1)(-1) - (-2y)(0) \div -1(\cancel{-1})(\cancel{-1}) - (-2y)(+2z)$$

$$= 1 \div -1 + (8xyz)$$

$$= \frac{-1}{-1 \mp 8xyz} = \frac{1}{1 + 8xyz}$$

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$$\frac{\partial y}{\partial u} = -\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,u,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}} \qquad \frac{\partial z}{\partial u} = -\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,u)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$$

$\frac{\partial f_1}{\partial x} = -1$; $\frac{\partial f_1}{\partial y} = -2y$; $\frac{\partial f_1}{\partial z} = 0$; $\frac{\partial f_2}{\partial x} = 0$; $\frac{\partial f_2}{\partial y} = -1$; $\frac{\partial f_2}{\partial z} = -2z$

$\frac{\partial f_3}{\partial x} = -2x$; $\frac{\partial f_3}{\partial y} = 0$; $\frac{\partial f_3}{\partial z} = -1$

$$= \begin{vmatrix} -1 & -2y & 0 \\ 0 & -1 & -2z \\ -2x & 0 & -1 \end{vmatrix} = \cancel{-1(-2z)(-2x) - (-2y)}$$
$$-1(-1)(-1) - (-2y)(-2x)(-2z)$$
$$= -1 + 8yxz$$

$\frac{\partial f_1}{\partial u} = 1$ $\quad \frac{\partial f_2}{\partial u} = 0$ $\quad \frac{\partial f_3}{\partial u} = 0$

$$= \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & -2z \\ -2x & 0 & -1 \end{vmatrix} = -1(0) - 1(-2x)(-2z)$$
$$= -4xz$$

$$\frac{\partial y}{\partial u} = \frac{-(-4xz)}{8yzx - 1} = \frac{4xz}{-(8yzx+1)}$$

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Q2. If u, v, w are roots of $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$ (λ),
find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Ans2. $= \lambda^3-x^3-3\lambda x(\lambda-x)+\lambda^3-y^3-3\lambda y(\lambda-y)+\lambda^3-z^3-3\lambda z(\lambda-z)$

$= \lambda^3-x^3-3\lambda^2x+3\lambda x^2+\lambda^3-y^3-3\lambda^2y+3\lambda y^2+\lambda^3-z^3-3\lambda^2 z+3\lambda z^2$

$= 3\lambda^3-x^3-y^3-z^3-3\lambda^2(x+y+z)+3\lambda(x^2+y^2+z^2)$

$= 3\lambda^3-3\lambda^2(x+y+z)+3\lambda(x^2+y^2+z^2)-(x^3+y^3+z^3)$

Sum of roots :- $u+v+w = \frac{-b}{a} = \frac{-(-3(x+y+z))}{3}$

$= x+y+z$

Products of roots :- $uv+vw+wu = \frac{c}{a} = \frac{3(x^2+y^2+z^2)}{3}$

$= x^2+y^2+z^2$

$uvw = \frac{-d}{a} = -\left(-\frac{(x^3+y^3+z^3)}{3}\right)$

$f_1(x,y,z,u,v,w) = u+v+w-(x+y+z)=0$

$f_2(x,y,z,u,v,w) = uv+vw+uw-(x^2+y^2+z^2)=0$

$f_3(x,y,z,u,v,w) = uvw-\frac{(x^3+y^3+z^3)}{3}$

$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^{n=3}\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$

$\frac{\partial f_1}{\partial x} = -1 \quad \frac{\partial f_1}{\partial y} = -1 \quad \frac{\partial f_1}{\partial z} = -1 \quad \frac{\partial f_2}{\partial x} = -2x \quad \frac{\partial f_2}{\partial y} = -2y \quad \frac{\partial f_2}{\partial z} = -2z$

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$\frac{\partial f_3}{\partial x} = -x^2$ $\frac{\partial f_3}{\partial y} = -y^2$ $\frac{\partial f_3}{\partial z} = -z^2$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= (-1)^3(2)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$C_1 = C_1 - C_2$ $C_2 = C_2 - C_3$

$$= (-1)^3(2)\begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$$

$$= -2\left[(x-y)(y^2-z^2) - (y-z)(x^2-y^2)\right]$$
$$= -2\left[(x-y)(y-z)(y+z) - (y-z)(x-y)(x+y)\right]$$

$$= -2\left[(x-y)(y-z)(z-x)\right]$$

$\frac{\partial f_1}{\partial u} = 1$ $\frac{\partial f_1}{\partial v} = 1$ $\frac{\partial f_1}{\partial w} = 1$ $\frac{\partial f_2}{\partial u} = v+w$ $\frac{\partial f_2}{\partial v} = u+w$ $\frac{\partial f_2}{\partial w} = v+u$

$\frac{\partial f_3}{\partial u} = vw$ $\frac{\partial f_3}{\partial v} = uw$ $\frac{\partial f_3}{\partial w} = uv$

$$\begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & v+u \\ vw & uw & uv \end{vmatrix}$$

$C_1 = C_1 - C_2$ $C_2 = C_2 - C_3$

$$= \begin{vmatrix} 0 & 0 & 1 \\ v-u & w-v & v+u \\ w(v-u) & u(w-v) & uv \end{vmatrix} = [u(v-u)(w-v)) - (w-v)(v-u)(w$$

$= (v-u)(w-v)(u-w)$

Putting in formula

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -\left[\frac{-2[(x-y)(y-z)(z-x)]}{(v-u)(w-v)(u-w)}\right]$$

$$= \frac{-2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

⇒ Total differentiation :-

$z = f(x,y)$

$$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

$$\boxed{\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}}$$

when $z=0$ / $z=$ constt.

$$dz = 0 \Rightarrow \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y} = \frac{-p}{q}} \;*$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-p}{q}\right) = -\left[\frac{q\cdot\frac{dp}{dx} - p\cdot\frac{dq}{dx}}{q^2}\right]$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)dy$$

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$$\frac{dP}{dx} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y \partial x}\frac{dy}{dx}$$

$$\frac{dP}{dx} = r + s\left(-p/q\right)$$

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy$$

$$dq = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)dy$$

$$dq = \frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy$$

$$\frac{dq}{dx} = s + t\left(\frac{dy}{dx}\right) = s + t\left(-p/q\right)$$

$$\frac{d^2y}{dx^2} = -\left[\frac{q \cdot \frac{dp}{dx} - p\frac{dq}{dx}}{q^2}\right] = -\left[\frac{q\left(r + s(-p/q)\right) - p\left(s + t(-p/q)\right)}{q^2}\right]$$

$$= -\left[\frac{qr - ps - ps + p^2t/q}{q^2}\right]$$

*

$$\boxed{\frac{d^2y}{dx^2} = -\left[\frac{p^2t - 2ps + q^2r}{q^3}\right]}$$

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Q1. $x^3 + 3x^2y + 6xy^2 + y^3 = 1$ find $\frac{dy}{dx}$

Ans. $f(x,y) = x^3 + 3x^2y + 6xy^2 + y^3 - 1 = 0$

$p = f_x = 3x^2 + 6xy + 6y^2.$

$q = f_y = 3x^2 + 6x(2y) + 3y^2$

$$\frac{dy}{dx} = -\frac{(3x^2 + 6xy + 6y^2)}{(3x^2 + 12xy + 3y^2)} = -\frac{(x^2 + 2xy + 2y^2)}{(x^2 + 4xy + y^2)}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{q^2r - 2pqs + p^2t}{q^3}\right]$$

$$r = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2 + 6xy + 6y^2\right)$$

$$= 6x + 6y = 6(x+y)$$

$$s = \frac{\partial^2 y}{\partial y \partial x} = \frac{\partial}{\partial y}(p) = \frac{\partial}{\partial y}\left(3x^2 + 6xy + 6y^2\right)$$

$$= 6x + 12y = 6(x+2y)$$

$$t = \frac{\partial^2 y}{\partial y^2} = \frac{\partial}{\partial y}(q) = 12x + 6y = 6(2x+y)$$

$$\frac{\partial^2 y}{dx^2} = -\left[\frac{(3x^2+12xy+3y^2)^2(6(x+y)) - 2(3x^2+6xy+6y^2)(6(x+\ldots + (3x^2+6xy+6y^2)^2(6(2x+y))}{(3x^2+12xy+3y^2)^3}\right]$$

$$= -\left[\frac{\{9x^4+144x^2y^2+9y^4+2(3x^2)(12xy)+2(12xy)(3y\ldots + 2(3x^2)(3y^2)\}(6(x+y)) - 36x^3 - 72x^2y - 72x^2y - 144x\ldots - 72xy^2 - 144y^3 + (9x^4+36x^2y^2+36y^4+36x^3y + 72xy^3+36y^2x)(12x+6y)}{(3x^2+12xy+3y^2)^3}\right]$$

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Q2. find $\frac{dy}{dx}$ when $(\cos x)^y = (\sin y)^x$

take $f(x,y) = (\cos x)^y - (\sin y)^x = 0$

$$\frac{dy}{dx} = -\frac{p}{q}$$

$$p = f_x = y(\cos x)^{y-1}.(-\sin x) - (\sin y)^x . \log_e(\sin y)$$

$$q = f_y = (\cos x)^y \log_e(\cos x) - x(\sin y)^{x-1}(\cos y)$$

$$\frac{dy}{dx} = \frac{-p}{q} = \frac{+y\sin x(\cos x)^{y-1} + (\sin y)^x \log_e(\sin y)}{(\cos x)^y \log_e \cos x - x(\sin y)^{x-1}(\cos y)}$$

$$= \frac{y(\cos x)^y \frac{(\sin x)}{\cos x} - (\sin y)^x \log(\sin y)}{(\cos x)^y \log(\cos x) - x(\sin y)^x \frac{\cos y}{\sin y}}$$

$(\cos x)^y = (\sin y)^x$ Put $(\sin y)^x = (\cos x)^y$

$$= \frac{y(\cos x)^y \frac{(\sin x)}{(\cos x)} - (\cos x)^y \log(\sin y)}{(\cos x)^y \log(\cos x) - x(\cos x)^y \frac{\cos y}{\sin y}}$$

$$= \frac{\cancel{(\cos x)^y}[y\tan x - \log\sin y]}{\cancel{(\cos x)^y}[\log(\cos x) - x\sec y]}$$

$$\boxed{\frac{dy}{dx} = \frac{y\tan x - \log\sin y}{\log(\cos x) - x\sec y}}$$

TAYLOR SERIES EXPANSION

If $f(x,y)$ & all its partial derivates upto nth order are finite & continuous, for all points x & y where x lies b/w a & $a+h$ & y lies b/w b & $b+k$

$$\forall \quad a \le x \le a+h$$
$$b \le y \le b+k$$

then taylor series expansion is given by

$x = a+h \Rightarrow \boxed{h = x-a}$; $y = b+k \Rightarrow \boxed{k = y-b}$

$$f(a+h, b+k) = f(x,y)$$
$$= f(a,b) + \frac{1}{1!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(x,y)\Big|_{(a,b)}$$
$$+ \frac{1}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(x,y)\Big|_{(a,b)}$$
$$+ \frac{1}{3!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^3 f(x,y)\Big|_{(a,b)} + \cdots$$

$$f(x,y) = f(a+h, b+k) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$

Q: $f(x,y) = e^x \sin y$, expand in powers of x & y

choose $a = b = 0$

$f(x,y) = e^x \sin y$ $\qquad f(0,0) = 0$

$f_x = e^x \sin y$	$f_x(0,0) = 0$
$f_y = e^x \cos y$	$f_y(0,0) = 1$
$f_{xx} = e^x \sin y$	$f_{xx}(0,0) = 0$
$f_{xy} = e^x \cos y$	$f_{xy}(0,0) = 1$
$f_{yy} = -e^x \sin y$	$f_{yy}(0,0) = 0$
$f_{xxx} = e^x \sin y$	$f_{xxx}(0,0) = 0$
$f_{xxy} = e^x \cos y$	$f_{xxy}(0,0) = +1$
$f_{xyy} = -e^x \sin y$	$f_{xyy}(0,0) = 0$
$f_{yyy} = -e^x \cos y$	$f_{yyy}(0,0) = -1$

$$f(x,y) = e^x \sin y = f(0,0) + \frac{1}{1!}\left[(x-0) f_x(0,0) + (y-0) f_y(0,0)\right]$$

$$+ \frac{1}{2!}\left[(x-0)^2 f_{xx}(0,0) + 2(x-0)(y-0) f_{xy}(0,0) + (y-0)^2 f_{yy}(0,0)\right]$$

$$+ \frac{1}{3!}\left[(x-0)^3 f_{xxx}(0,0) + 3(x-0)^2(y-0) f_{xxy}(0,0) + 3(x-0)(y-0)^2 f_{xyy}(0,0) + (y-0)^3 f_{yyy}(0,0)\right] + \cdots$$

$$f(x,y) = e^x \sin y = [y] + \frac{1}{2}[2xy] + \frac{1}{6}[3x^2y - y^3] + \cdots$$

Ans.

Q2. Expand $f(x,y) = e^x \cos y$ near the pt $(1, \pi/4)$

$a = 1, \; b = \pi/4$

$f(x,y) = e^x \cos y$	$f(1,\pi/4) = e/\sqrt{2}$
$f_x = e^x \cos y$	$f_x(1,\pi/4) = e/\sqrt{2}$
$f_y = -e^x \sin y$	$f_y(1,\pi/4) = -e/\sqrt{2}$
$f_{xx} = e^x \cos y$	$f_{xx}(1,\pi/4) = e/\sqrt{2}$
$f_{xy} = -e^x \sin y$	$f_{xy}(1,\pi/4) = -e/\sqrt{2}$
$f_{yy} = -e^x \cos y$	$f_{yy}(1,\pi/4) = -e/\sqrt{2}$
$f_{xxx} = e^x \cos y$	$f_{xxx}(1,\pi/4) = e/\sqrt{2}$
$f_{xxy} = -e^x \sin y$	$f_{xxy}(1,\pi/4) = -e/\sqrt{2}$
$f_{xyy} = -e^x \cos y$	$f_{xyy}(1,\pi/4) = -e/\sqrt{2}$
$f_{yyy} = e^x \sin y$	$f_{yyy}(1,\pi/4) = e/\sqrt{2}$

$$f(x,y) = e^x \cos y = f(1,\pi/4) + \frac{1}{1!}\Big[(x-1)^2 f_x(1,\pi/4) + (y-\pi/4) f_y(1,\pi/4)\Big]$$

$$+ \frac{1}{2!}\Big[(x-1)^2 f_{xx}(1,\pi/4) + 2(x-1)(y-\pi/4) f_{xy}(1,\pi/4) + (y-\pi/4)^2 f_{yy}(1,\pi/4)\Big]$$

$$+ \frac{1}{3!}\Big[(x-1)^3 f_{xxx}(1,\pi/4) + 3(x-1)^2(y-\pi/4) f_{xxy}(1,\pi/4) + 3(x-1)(y-\pi/4)^2 f_{xyy}(1,\pi/4)$$
$$+ (y-\pi/4)^3 f_{yyy}(1,\pi/4)\Big]$$

$$f(x,y) = e^x \cos y = \frac{e}{\sqrt{2}} + 1\left[(x-1)\frac{e}{\sqrt{2}} \mp \left(y-\frac{\pi}{4}\right)\frac{e}{\sqrt{2}}\right] + \frac{1}{2}\left[(x-1)^2\frac{e}{\sqrt{2}} + 2(x-1)\left(y-\frac{\pi}{4}\right)\left(\frac{-e}{\sqrt{2}}\right)\right.$$
$$\left. + \left(y-\frac{\pi}{4}\right)^2\left(\frac{-e}{\sqrt{2}}\right)\right] + \frac{1}{6}\left[(x-1)^3\frac{e}{\sqrt{2}} - \frac{e}{\sqrt{2}}(3)(x-1)^2\left(y-\frac{\pi}{4}\right) - \frac{e}{\sqrt{2}}(3)(x-1)\left(y-\frac{\pi}{4}\right)^2\right.$$
$$\left. + \left(y-\frac{\pi}{4}\right)^3\frac{e}{\sqrt{2}}\right] \quad \underline{\text{Ans.}}$$

Date :- 1 Dec 2022

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Q1. $f(x,y) = \tan^{-1}(xy)$ using Taylor series expansion & hence find $f(0.9, -1.2)$ Take $a=1, b=-1$; $h = 0.9-1 = -0.1$, $k = -1.2-(-1) = -0.2$

A1.
$f(x,y) = \tan^{-1}(x,y)$ $\quad$ $\cancel{a=b}=0$

$f_x = y/1+(xy)^2$

$f_y = x/1+(xy)^2$

$f_{xx} = \dfrac{(1+xy)^2(0)-(y)(2xy^2)}{(1+(xy)^2)^2}$

$f_{xy} =$

$f_{yy} =$

$f_{xy} = \dfrac{(1+(xy)^2)-(x)(2xy^2)}{(1+(xy)^2)^2}$

$f_{yy} = \dfrac{-x(2yx^2)}{(1+(xy)^2)}$

$f(1,-1) = \tan^{-1}(-1) = -\pi/4$

$f_x(1,-1) = -1/2$

$f_y(1,-1) = 1/2$

$f_{xx}(1,-1) = \dfrac{-2xy^3}{(1+(xy)^2)^2} = \dfrac{2}{4} = \dfrac{1}{2}$

$f_{xy} = \dfrac{2-2}{(1+(xy))^2} = 0$

$f_{yy} = \dfrac{2}{4} = \dfrac{1}{2}$

$$f(x,y) = \tan^{-1}(xy) = f(1,-1) + \frac{1}{1!}\left[(x-1)f_x(1,-1) + (y+1)f_y(1,-1)\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,-1) + 2(x-1)(y+1)f_{xy}(1,-1) + (y+1)^2 f_{yy}(1,-1)\right] + \cdots\infty$$

$$f(x,y) = \tan^{-1}(xy) = \frac{-\pi}{4} + \left[(x-1)\left(\frac{-1}{2}\right) + (y+1)\left(\frac{1}{2}\right)\right]$$
$$+ \frac{1}{2}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y+1)(0) + (y+1)^2\frac{1}{2}\right] + \cdots\infty$$

$\cancel{f(x,y) = -\frac{\pi}{4} + \frac{x}{2}}$

$$f(0.9,-1.2) = -\frac{\pi}{4} + \left[(-0.1)\left(\frac{-1}{2}\right) + (-0.2)\left(\frac{1}{2}\right)\right]$$
$$+ \frac{1}{2}\left[(-0.1)^2\left(\frac{1}{2}\right) + (-0.2)^2\left(\frac{1}{2}\right)\right]$$

$$= -\frac{\pi}{4} + \frac{0.1}{2} - \frac{0.2}{2} + \frac{0.01}{4} + \frac{0.04}{4}$$

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$$= -\frac{\pi}{4} + \frac{0.2 - 0.4 + 0.01 + 0.04}{4}$$

$$= \frac{-\pi - 0.2 + 0.05}{4} = -\frac{3.14285 - 0.2 + 0.05}{4}$$

$$= -\frac{3.34285 + 0.05}{4} = -\frac{3.29285}{4}$$

$$= -0.8232 \text{ Ans.}$$

(4-6 marks)

Q2. Expand $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ about $(1,1)$ uptill 2nd degree

(H.W)

LAGRANGES METHOD OF MULTIPLIERS..

$f(x,y,z) = 0 \rightarrow$ ft to be max / min

$\phi(x,y,z) = 0 \rightarrow$ defines relationship b/w (x,y,z)

λ (Lagranges multiplier) $\neq 0$

$$\psi = f + \lambda\phi = 0$$

$$\left.\begin{aligned} \frac{\partial \psi}{\partial x} &= \frac{\partial f}{\partial x} + \lambda\frac{d\phi}{\partial x} = 0 \\ \frac{\partial \psi}{\partial y} &= \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \\ \frac{\partial \psi}{\partial z} &= \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \\ \phi(x,y,z) &= 0 \end{aligned}\right\}$$

Lagranges Eqns

Solve for x, y, z & λ.

gives only stationary pt (maximum or minimum)

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Q. Find the maximum value of
$u = x^p y^q z^r$ when the variables x, y, z are subjected to the conditions $ax+by+cz = p+q+r$.

Ans. $\phi = ax+by+cz-p-q-r=0$

$\log u = p\log x + q\log y + r\log z$

$u + \lambda\phi = 0$

$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{p}{x} \Rightarrow \frac{\partial u}{\partial x} = \frac{pu}{x}$ | $\frac{\partial \phi}{\partial x} = a$

$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{q}{y} \Rightarrow \frac{\partial u}{\partial y} = \frac{qu}{y}$ | $\frac{\partial \phi}{\partial y} = b$

$\frac{1}{u}\frac{\partial u}{\partial z} = \frac{r}{z} \Rightarrow \frac{\partial u}{\partial z} = \frac{ru}{z}$ | $\frac{\partial \phi}{\partial z} = c$

(i) eqn

$\frac{pu}{x} + \lambda a = 0$ $\quad (\times x) \rightarrow pu + \lambda a x = 0 \Rightarrow \frac{-pu}{\lambda} = ax$

(ii) $\frac{qu}{y} + \lambda(b) = 0$ $\quad (\times y) \rightarrow qu + \lambda by = 0 \Rightarrow \frac{-qu}{\lambda} = by$

(iii) $\frac{ru}{z} + \lambda c = 0$ $\quad (\times z) \rightarrow ru + \lambda cz = 0 \Rightarrow \frac{-ru}{\lambda} = cz$

(iv) $ax+by+cz-p-q-r=0$

$\frac{-pu}{\lambda} - \frac{qu}{\lambda} - \frac{ru}{\lambda} - p - q - r = 0$

$-u(p+q+r) - \lambda(p+q+r) = 0$

$\boxed{u = -\lambda} \rightarrow \boxed{\lambda = -u}$

$\frac{-pu}{\lambda} = ax \Rightarrow \boxed{\frac{p}{a} = x}$ lly $\boxed{\frac{q}{b} = y}$, $\boxed{\frac{r}{c} = z}$

$u(x,y,z) = \left(\frac{p}{a}\right)^p \left(\frac{q}{b}\right)^q \left(\frac{r}{c}\right)^r$

HWQ2. $f(x,y) = \tan^{-1}(y/x)$

$$fx = \frac{1}{1+(y/x)^2} \times \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$fy = \frac{1}{1+(y/x)^2} \times \frac{1}{x} = \frac{1}{x+y^2/x} = \frac{1}{x+xy^2/x^2} = \frac{1}{(1+y^2/x^2)x}$$

$$fxx = \frac{-(-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$fxy = \frac{-(1-y^2/x^2)}{(x+y^2/x)^2} = \frac{y^2/x^2-1}{(x+y^2/x)^2}$$

$$fyy = \frac{-2y/x}{(x+y^2/x)^2}$$

$f(0,0) = \frac{\pi}{4}$; $fx(1,1) = \frac{-1}{2}$; $fy(1,1) = \frac{1}{2}$; $fxx(1,1) = \frac{1}{2}$

$fxy(1,1) = 0$; $fyy(1,1) = \frac{-1}{2}$

$$f(x,y) = \frac{\pi}{4} + 1\left((x-1)\frac{-1}{2} + (y-1)\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}(x-1)^2 + 2(x-1)(y-1)(0) - \frac{1}{2}(y-1)^2\right) + \cdots$$

$$f(x,y) = \frac{\pi}{4} + -\frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \cdots$$

Ans.

formula of taylor series

$$f(x,y) = f(a,b) + \frac{1}{1!}\left((x-a)\,fx(a,b) + (y-b)\,fy(a,b)\right) + \frac{1}{2!}\left((x-a)^2 fxx(a,b) + 2(x-a)(y-b)\,fxy(a,b) + (y-b)^2 fyy(a,b)\right) + \cdots$$

Q2. (HW) A rectangular box opened at top has capacity of 256 cubic feet. Determine the dimensions of the box such that the least material is req for the construction of the box. Use Lagranges method of multipliers to solve the ques.

A2. Surface Area $= lb + 2(l+b)h = A$ (function to be min)

$V = lbh - 256 = 0$

$\psi = A + \lambda V$

$\frac{\partial A}{\partial l} = b + 2h$	$\frac{\partial V}{\partial l} = bh$
$\frac{\partial A}{\partial b} = l + 2h$	$\frac{\partial V}{\partial b} = lh$
$\frac{\partial A}{\partial h} = 2l + 2b$	$\frac{\partial V}{\partial h} = lb$

$\frac{\partial \psi}{\partial x} = b + 2h + \lambda bh = 0 \; (\times l) \rightarrow lb + 2lh + \lambda lbh = 0$

$\frac{\partial \psi}{\partial y} = l + 2h + \lambda lh = 0 \; (\times b) \Rightarrow lb + 2hb + \lambda lbh = 0$

$\frac{\partial \psi}{\partial z} = 2l + 2b + \lambda lb = 0 \; (\times h) \Rightarrow 2lh + 2bh + \lambda lbh = 0$

$lbh - 256 = 0 \; (\times \lambda) \Rightarrow \lambda lbh - 256\lambda = 0$

$\lambda lbh = 256\lambda$

$lb + 2lh + 256\lambda = 0 \Rightarrow lb + 2lh = -256\lambda$ — (i)

$lb + 2hb + 256\lambda = 0 \Rightarrow lb + 2hb = -256\lambda$ — (ii)

$2h(l+b) + 256\lambda = 0 \Rightarrow 2h(l+b) = -256\lambda$ — (iii)

(i) = (ii)	(ii) = (iii)
$l = b$	$b = 2h$

$V = lbh = 256$

$= \cancel{l \times l} \times 2h \times 2h \times h = 256$

$h^3 = \frac{256}{4} = 64$

$\boxed{h = 4 \text{ units}}$ $\boxed{b = 8 \text{ units} = l}$

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Q1. If $u = ax^2 + by^2 + cz^2$ where $x^2+y^2+z^2=1$, $lx+my+nz$, prove that stationary of values of 'u' satisfy the eqn:-

$$\frac{l^2}{a-u} + \frac{m^2}{b-u} + \frac{n^2}{c-u} = 0$$

A1. $\phi_1 = x^2+y^2+z^2-1 = 0$
$\phi_2 = lx+my+nz = 0$

$$\psi = u + \lambda_1\phi_1 + \lambda_2\phi_2$$

$$\frac{\partial\psi}{\partial x} = \frac{\partial u}{\partial x} + \lambda_1\frac{\partial\phi_1}{\partial x} + \lambda_2\frac{\partial\phi_2}{\partial x} = 2ax + 2x\lambda_1 + l\lambda_2 = 0 \quad (\times x)$$

$$\frac{\partial\psi}{\partial y} = \frac{\partial u}{\partial y} + \lambda_1\frac{\partial\phi_1}{\partial y} + \lambda_2\frac{\partial\phi_2}{\partial y} = 2by + \lambda_1 2y + \lambda_2 m = 0 \quad (\times y)$$

$$\frac{\partial\psi}{\partial z} = \frac{\partial u}{\partial z} + \lambda_1\frac{\partial\phi_1}{\partial z} + \lambda_2\frac{\partial\phi_2}{\partial z} = 2cz + \lambda_1 2z + \lambda_2 n = 0 \quad (\times z)$$

(i) $2ax^2 + 2x^2\lambda_1 + xl\lambda_2 = 0$
(ii) $2by^2 + 2y^2\lambda_1 + my\lambda_2 = 0$
(iii) $2cz^2 + 2z^2\lambda_1 + nz\lambda_2 = 0$

adding all we get,

$$2(ax^2+by^2+cz^2) + 2\lambda_1\underbrace{(x^2+y^2+z^2)}_{1} + \lambda_2\underbrace{(xl+my+nz)}_{0} = 0$$

$$2u + 2\lambda_1 = 0 \Rightarrow \boxed{-u = \lambda_1}$$

$$2ax^2 + 2x^2(-u) + xl\lambda_2 = 0$$
$$2x^2(a-u) = -xl\lambda_2$$
$$x = \frac{-l\lambda_2}{2(a-u)}$$

||ly $y = \dfrac{-\lambda_2 m}{2(b-u)}$; $z = \dfrac{-\lambda_2 n}{2(c-u)}$

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Putting values of x, y & z in $lx+my+nz=0$

$$l\left(\frac{-\lambda_2 l}{2(a-u)}\right)+m\left(-\frac{\lambda_2 m}{2(b-u)}\right)+n\left(\frac{-\lambda_2 n}{2(c-u)}\right)=0$$

$$\frac{-\cancel{\lambda_2} l^2}{\cancel{2}(a-u)}+\frac{(-\cancel{\lambda_2})m^2}{\cancel{2}(b-u)}+\frac{n^2(-\cancel{\lambda_2})}{\cancel{2}(c-u)}=0$$

$$\frac{l^2}{a-u}+\frac{m^2}{b-u}+\frac{n^2}{c-u}=0$$

ERROR APPROXIMATIONS :-

$y=f(x)$

$$\frac{\partial y}{\partial x}\cong\frac{dy}{dx}\Rightarrow \partial y=\frac{dy}{dx}\partial x$$

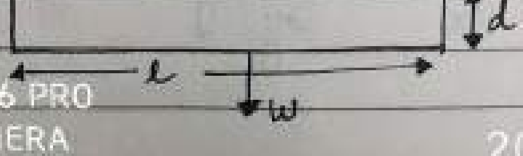
$$\boxed{\delta y\cong\frac{dy}{dx}\delta x}$$

$\delta x=$ absolute error in x

$\frac{\delta x}{x}=$ Relative error in x

$$\frac{\delta x}{x}\times 100=\%\text{ error in }x$$

Q. The deflection at the center of ~~mass~~ a rod of length (l) and diameter (d) supported at its ends and ~~bonded~~ loaded at the centre with a weight (w) varies $D\propto wl^3d^{-4}$. What is % increase in the deflection corresponding to the % increase in w, l & D is 3, 2 & 1 respectively.



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Ans) $D \propto w l^3 d^{-4}$

$D = K w l^3 d^{-4}$

$\log D = \log K + \log w + 3\log l - 4\log d$

$\frac{1}{D}\delta D = 0 + \frac{1}{w}\delta w + \frac{3}{l}\delta l - \frac{4}{d}\delta d$

Multiply by 100

$\frac{\delta D}{D}\times 100 = \frac{\delta w}{w}\times 100 + 3\frac{\delta l}{l}\times 100 - 4\frac{\delta d}{d}\times 100$

$\frac{\delta D}{D}\times 100 = 3 + (3)2 - (4)(1)$

$= 5$

$\left[\frac{\delta w}{w} = 3 ; \frac{\delta l}{l} = 2 ; \frac{\delta d}{d} = 1\right]$

$$\boxed{\frac{\delta D}{D}\times 100 = 5\%}$$

Q. Find $[(3.82)^2 + 2(2.1)^3]^{1/5}$ error

Take a ft in x & y

$f(x,y) = [x^2 + 2y^3]^{1/5}$

choose nearest int round off values

$x = 4,\ y = 2$

$\delta x = 3.82 - 4 = 0.18$

$\delta y = 2.1 - 2 = 0.1$

$f(4,2) = [16 + 16]^{1/5} = 2$

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

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$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,2)} = \left.\left\{\frac{1}{5}\left[x^2+2y^3\right]^{-4/5} \times 2x\right\}\right|_{(4,2)}$$

$$= \frac{1}{5}(32)^{-4/5} \times 8$$

$$= \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,2)} = \left.\left\{\frac{1}{5}\left[x^2+2y^3\right]^{-4/5} \times 6y^2\right\}\right|_{(4,2)}$$

$$= \frac{1}{5}(2)^{-4} \times 6(4) = \frac{3}{10}$$

$$\delta f = \frac{1}{10} \times (-0.18) + \frac{3}{10}(0.1)$$

$$= -0.018 + 0.03$$

$$\boxed{\delta f = 0.012}$$

$$f(3.82, 2.1) \equiv f(4,2) + \delta f = 2 + 0.012$$
$$= 2.012$$

Q. Find $(1.04)^{3.01}$ error

$f(x,y) = (x)^y$

$x=1,\ y=3$

$\delta x = 1.04 - 1 = 0.04$

$\delta y = 0.01$

$f(1,3) = 1$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,3)} = yx^{y-1} = 3\times 1 = 3$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,3)} = x^y \log_y x = 1 \log 1 = 0$$

$$\delta f = 3\times 0.04 + 0\times 0.01$$
$$= 0.12$$

$$f(1.04, 3.01) = f(1,3) + \delta f$$
$$= 1 + 0.12 = 1.12$$

Date :- 6 Dec, 2022

MAXIMA & MINIMA

$$z = f(x,y)$$

(i) To find stationary pts

(a) $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y} \rightarrow$ Set of points $P(x,y)$.

(b) $\left. rt - s^2 \right|_{P(x,y)} > 0$

$$\left.\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right|_{P(x,y)} > 0$$

Then $P(x,y)$ is a stationary pt.

(ii) To find Maxima & Minima

$\frac{\partial^2 f}{\partial x^2} / \frac{\partial^2 f}{\partial y^2} < 0$ at $P(x,y)$

then $P(x,y)$ is pt of Maxima

$\frac{\partial^2 f}{\partial x^2} / \frac{\partial^2 f}{\partial y^2} > 0$ at $P(x,y)$

then $P(x,y)$ is pt of Minima

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Q1. $f(x,y) = x^3+y^3-63(x+y)+12xy$

(i) $\frac{\partial f}{\partial x} = 0 = 3x^2 - 63 + 12y$ — (i)

$\frac{\partial f}{\partial y} = 3y^2 - 63 + 12x = 0$ — (ii)

$3y^2+12x = 63$

$x = \frac{63-3y^2}{12}$ — (ii)

(ii) in (i)

$3\left(\frac{63-3y^2}{12}\right)^2 - 63 + 12y = 0$

$3[(63)^2 + (3y^2)^2 - 2(63)(3y^2)] - 63(12)^2 + (12)^3 y = 0$

$3[3969 + 9y^4 - 378y^2] - 9072 + 1728y = 0$

$11{,}907 + 27y^4 - 1134y^2 - 9072 + 1728y = 0$

$27y^4 + 2835 - 1134y^2 + 1728y = 0$

Mthd-2

Subtract (i) & (iii)

$3x^2 - \cancel{63} + 12y - 3y^2 + \cancel{63} - 12x = 0$

$3x(x-4) - 3y(y-4) = 0$

$3x(x-4) = 3y(y-4)$

$(-1,5), (5,-1), (3,3), (-7,-7)$ are some sets

(ii) $r = \frac{\partial^2 f}{\partial x^2} = 6x$ $\quad t = \frac{\partial^2 f}{\partial y^2} = 6y$ $\quad s = \frac{\partial^2 f}{\partial x \partial y} = 12$

$63 \times 63 = 189 + 3780 = 3969$

$63 \times 6 = 378$

$144 \times 63 = 432 + 8640 = 9072$

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$rt - s^2 = (6x)(6y) - (12)^2$

$(-1,5)$

$6(-1)6(5) - (12)^2$

$-180 - 144 < 0$

$(5,-1) < 0 \quad (-ve)$

$(3,3) > 0 \quad (+ve)$

$(-7,-7) > 0 \quad (+ve)$

Only $(3,3)$ & $(-7,-7)$ are stationary pts.

$r \left| \begin{array}{l} (3,3) = +ve \\ (-7,-7) = -ve \end{array} \right.$

$(3,3)$ is pt of minima & $(-7,-7)$ is pt of Max.

$$\text{Max value} = (-7)^3 + (-7)^3 - 63(-7-7) + 12(-7)(-7)$$
$$= -343 - 343 - 63(-14) + 12(49)$$
$$= -686 + 882 + 588$$
$$= -686 + 1470$$
$$= 784$$

Min value $= -216$

63
14
252
630
882

49
12
98
490
588

588
882
1470
-686
784

Q2. The Alt. of Right circular cone is 15cm & is increasing at 0.2 cm/sec. The radius of the base is 10cm & is decreasing at 0.3cm/sec. How fast the volume changes

A2. $V = \frac{1}{3}\pi r^2 h$

$h = 15\,cm \qquad \frac{dh}{dt} = 0.2\,\frac{cm}{sec}$

$r = 10\,cm \qquad \frac{dr}{dt} = -0.3\,\frac{cm}{sec}$

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}$$

$x = h,\ y = r$

$$\frac{dv}{dt} = \frac{\partial v}{\partial h}\Big|_{(r,h)}\frac{dh}{dt} + \frac{\partial v}{\partial r}\Big|_{(r,h)}\frac{dr}{dt}$$

$$\frac{\partial v}{\partial h} = \frac{1}{3}\pi r^2 \qquad \frac{\partial v}{\partial r} = \frac{1}{3}\pi(2r)(h)$$

$$\frac{\partial v}{\partial h}\Big|_{(h=15,\,r=10)} = \frac{1}{3}\times\pi\times10\times10 \qquad \frac{\partial v}{\partial r} = \frac{1}{3}\times 3.14\times2(10)(15)$$

$$= 104.66 \qquad = 314.28$$

$$= \frac{100}{3}\pi \qquad = 100\pi$$

$$\frac{dv}{dt} = \frac{100\pi}{3}(0.2) + 100\pi(-0.3)$$

$$= -\frac{70\pi}{3}$$

Q3. Find the rate at which area of rectangle is incr. at a given instant when the sides of rectangle are 4ft (l) & 3ft (b) & are inc. at the rate of 1.5ft/sec (l) & 0.5ft/sec (b) respectively.

A3. $A = l \times b$

$$\frac{dA}{dt} = \frac{\partial A}{\partial l}\frac{dl}{dt} + \frac{\partial A}{\partial b}\frac{db}{dt} = 3(1.5) + 4(0.5)$$

$$\frac{\partial A}{\partial l} = b = 3 \qquad \frac{\partial A}{\partial b} = l = 4 \qquad = 4.5 + 2 = 6.5\,\frac{ft}{sec}$$

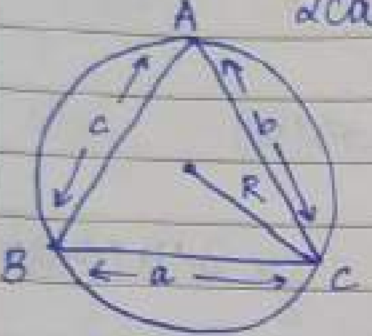
for (2-3 marks ques)

Q4. If the sides of a Δ a, b, c vary in such a way that the circumradius remains constant. prove that

$$\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=0$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc} \qquad \cos C=\frac{a^2+b^2-c^2}{2ab}$$

$$\cos B=\frac{c^2+a^2-b^2}{2ca}$$



$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

$$a=2R\sin A$$
$$b=2R\sin B$$
$$c=2R\sin C$$

$$da=2R\cos A\,dA$$
$$db=2R\cos B\,dB$$
$$dc=2R\cos C\,dC$$

$$\frac{da}{\cos A}=2R\,dA \qquad \frac{db}{\cos B}=2R\,dB \qquad \frac{dc}{\cos C}=2R\,dC$$

$$\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=2R\,(dA+dB+dC)=0$$

$$\angle A+\angle B+\angle C=180^\circ$$
$$dA+dB+dC=0$$

Q5. A rectangular box opened at top is to have a vol of 32 cubic cm. find dimensions of the box requiring least material for its construction

A5. $V = lbh = 32$

$h = \frac{32}{lb}$

$S = lb + 2h(l+b)$

$S = lb + 2\left(\frac{32}{lb}\right)(l+b)$

$\frac{\partial s}{\partial l} = b + \frac{64}{b}\left(\frac{-1}{l^2}\right)(l+b) + \frac{64}{lb}(1)$

$\frac{\partial s}{\partial b} = l + \frac{64}{l}\left(\frac{-1}{b^2}\right)(l+b) + \frac{64}{lb}(1)$

$\frac{\partial s}{\partial l} = b - \frac{64}{l^2 b}(l+b) + \frac{64}{lb} = 0 \quad \boxed{\times l^2 b^2} \quad -(i)$

$\frac{\partial s}{\partial b} = l - \frac{64}{lb^2}(l+b) + \frac{64}{lb} = 0 \quad \boxed{\times l^2 b^2} - (ii)$

(i) $= l^2 b^3 - 64b(l+b) + 64(lb) = 0$

$l^2 b^3 - 64lb - 64b^2 + 64lb = 0$

$l^2 b^3 + \cancel{128lb} - 64b^2 = 0 \Rightarrow l^2 b^3 = 64b^2$

$\boxed{b = 64/l^2}$

(ii) $= l^3 b^2 - 64l(l+b) + 64(lb) = 0$

$l^3 b^2 - 64l^2 - 64lb + 64lb = 0$

$l^3 b^2 = 64 l^2$

$\boxed{l = \frac{64}{b^2}} \Rightarrow l = \frac{64}{\left(\frac{64}{l^2}\right)^2} = 64 \times \frac{l^4}{(64)^2}$

$64 = l^3 \Rightarrow \boxed{l = 4}$

$\boxed{b = 4}$

$$rt - s^2\Big|_{(4,4)}^{(4,4)} > 0$$

$$\left(\frac{\partial^2 S}{\partial l^2}\right)\left(\frac{\partial^2 S}{\partial b^2}\right) - \left(\frac{\partial^2 S}{\partial l \partial b}\right)\Big|_{(4,4)}$$

$$\frac{\partial^2 S}{\partial l^2} = \frac{-64}{b}(-2l^{-3})(l+b) + \cancel{\frac{64}{l^2 b}} \cancel{-\frac{64}{l^2 b}}$$

$$= \frac{128\cancel{l}}{l^3 b}(l+b)\Big|_{(4,4)} = \frac{128}{(4)^4}(8) = \frac{64 \times 2 \times 8}{4^3 \times 4} = 4$$

$$\frac{\partial^2 S}{\partial b^2} = \frac{-64}{l}(-2b^{-3})(l+b) = 4$$

$$\frac{\partial^2 S}{\partial l \partial b} = 1 - \frac{64}{b^2}\left(\frac{-1}{l^2}\right)(l+b) - \frac{64}{lb^2}(1) - \frac{64}{l^2 b}$$

$$= 1 + \frac{64}{l^2 b^2}(l+b) - \frac{64}{lb^2} - \frac{64}{l^2 b} -$$

$$= 1 + \frac{64}{(4)^2(4)^2}(8) - \frac{64}{(4)^3} - \frac{64}{(4^3)}$$

$$= 1 + 2 - 1 - 1 = 1$$

Hence $\left(\frac{\partial^2 S}{\partial l^2}\right)\left(\frac{\partial^2 S}{\partial b^2}\right) - \left(\frac{\partial^2 S}{\partial l \partial b}\right)\Big|_{(4,4)} > 0$

Since we have only one point, hence its pt of minima

$l = b = 4\,cm$

$h = \frac{32}{4 \times 4} = 2\,cm$

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FORMULAS :-

(i) ~~term~~ Area of ellipse $= \pi ab$

(ii) Surface of a sphere $= 4\pi r^2$

(iii) Volume of a cylinder $= \pi r^2 h$

(iv) Volume of a sphere $= 4/3 \pi r^3$

(v) Volume of a cone $= \frac{1}{3} \pi r^2 h$

(vi) Volume of pyramid $= 1/3 \times$ area of base $\times h$

(vii) Volume of prism $=$ area of base $\times h$

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Extra Ques

Q1. $u = u\left(\dfrac{y-x}{xy}, \dfrac{z-x}{zx}\right)$

Find $x^2\dfrac{\partial u}{\partial x} + y^2\dfrac{\partial u}{\partial y} + z^2\dfrac{\partial u}{\partial z}$

A1. $u = u(r, s)$ where $r = \dfrac{y-x}{xy}$; $s = \dfrac{z-x}{zx}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} \quad ; \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\left(\frac{(-1)(xy)-(y-x)y}{x^2y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{(-1)(zx)-z(z-x)}{x^2z^2}\right)$$

$$= \frac{\partial u}{\partial r}\cdot\left(\frac{-y^2}{x^2y^2}\right) - \frac{\partial u}{\partial s}\left(\frac{z^2}{x^2z^2}\right)$$

$$= -\frac{\partial u}{\partial r}\cdot\left(\frac{1}{x^2}\right) - \frac{\partial u}{\partial s}\left(\frac{1}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\left(\frac{1(xy)-(x)(y-x)}{x^2y^2}\right) = +\frac{\partial u}{\partial r}\frac{x^2}{x^2y^2} = +\frac{\partial u}{\partial r}\left(\frac{1}{y^2}\right)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\cdot\left(\frac{(1)(zx)-(z-x)(x)}{z^2x^2}\right) = \frac{\partial u}{\partial s}\cdot\left(\frac{1}{z^2}\right)$$

$$x^2\left(\frac{\partial u}{\partial x}\right) + y^2\left(\frac{\partial u}{\partial y}\right) + z^2\left(\frac{\partial u}{\partial z}\right)$$

$$= x^2\left(-\frac{\partial u}{\partial r}\left(\frac{1}{x^2}\right) - \frac{\partial u}{\partial s}\left(\frac{1}{x^2}\right)\right) + y^2\left(\frac{\partial u}{\partial r}\left(\frac{1}{y^2}\right)\right) + z^2\left(\frac{\partial u}{\partial z}\left(\frac{1}{z^2}\right)\right)$$

$= 0$ Ans.

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Q2. If $u = f(r)$, $x = r\cos\theta$, $y = r\sin\theta$

Prove $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f(r)$

~~$x = r\cos\theta \Rightarrow r = x\sec\theta$; $y = r\sin\theta \Rightarrow r = y\,\mathrm{cosec}\,\theta$~~

~~$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} = f'(r) \cdot \sec\theta$~~

$x = r\cos\theta$, $y = r\sin\theta \Rightarrow x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$

$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2}x)$

$\frac{\partial u}{\partial x} = f'(r) \cdot x \cdot (x^2+y^2)^{-1/2}$

$\frac{\partial^2 u}{\partial x^2} = f''(r) \cdot x^2 (x^2+y^2)^{\cancel{-1/2}} + f(r) -\frac{1}{2}(x^2+y^2)^{-3/2} \cdot x \cdot (2x)$

$+ f(r) \cdot (x^2+y^2)^{-1/2}$

$\frac{\partial^2 u}{\partial x^2} = f''(r) \cdot x^2 (x^2+y^2)^{\cancel{-1/2}} + \frac{f'(r)}{\cancel{2}}(x^2+y^2)^{-3/2}(\cancel{2}x) \cdot x$

$+ f'(r)(x^2+y^2)^{-1/2}$

$= f''(r) \cdot x^2 (x^2+y^2)^{\cancel{-1/2}} \mp f'(r)(x^2+y^2)^{-3/2} \cdot x^2 + f'(r)(x^2+y^2)^{1/2}$

$\parallel$ ly $\frac{\partial^2 u}{\partial y^2} = f''(r) \cdot y^2 (x^2+y^2)^{\cancel{-1/2}} \mp f'(r)(x^2+y^2)^{-3/2} \cdot y^2 + f'(r)(x^2+y^2)^{1/2}$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{x^2+y^2}{\sqrt{x^2+y^2}}\right) + f'(r)\left(\frac{2}{\sqrt{x^2+y^2}} - \frac{x^2}{(\sqrt{x^2+y^2})^3} - \frac{y^2}{(\sqrt{x^2+y^2})^3}\right)$

$= f''(r)$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f(r).$

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Q3. Determine Max & minimum

$f(x,y) = \sin x + \sin y + \sin(x+y)$

A3. Let $z = f(x,y)$

$$\frac{\partial z}{\partial x} = \cos x + \cos(x+y) = 0 \quad -(i)$$

$$\frac{\partial z}{\partial y} = \cos y + \cos(x+y) = 0 \quad -(ii)$$

(i)-(ii)

$\cos x - \cos y = 0$

$\cos x = \cos y$

$x = 2n\pi \pm y \quad ; n \in I, \ y \in [0, \pi]$

$y = 0, \ x = 2\pi$ $\quad$ $y = 0, x = 0$

$y = \pi ; \ x = \pi$ $\quad$ $y = \frac{\pi}{2} ; \ x = \frac{5\pi}{2}$

$y = \pi ; \ x = 3\pi$

$\vdots \quad \vdots$

$y = \pi/2, \ x = 3\pi/2$

$rt - s^2 > 0$ $\quad$ $y = x = \pi/3$

$$r = \frac{\partial^2 z}{\partial x^2} = -\sin x - \sin(x+y)$$

$$\frac{\partial^2 z}{\partial y^2} = t = -\sin y - \sin(x+y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = s = -\sin(x+y)$$

$$(-\sin x - \sin(x+y))(-\sin y - \sin(x+y)) - (-\sin(x+y))^2 > 0$$

$x = 0, y = 0$

$$(-\sin(0) - \sin(0))(-\sin 0 - \sin 0) - (-\sin(0))^2 \not> 0$$

$x = 2\pi, y = 0 \not> 0$

$x = 5\pi/2, \ y = \pi/2$

$$\left(-\sin\frac{5\pi}{2} - \sin\left(\frac{5\pi}{2} + \frac{\pi}{2}\right)\right)\left(-\sin\frac{\pi}{2} - \sin\left(\frac{\pi}{2} + \frac{5\pi}{2}\right)\right)$$

$$- \left(-\sin\left(\frac{\pi}{2} + \frac{5\pi}{2}\right)\right)^2 \not> 0$$

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$(-1)(+1) = +1 \not> 0$

$y = \pi/2 \;,\; x = 3\pi/2$

$\left(-\sin\left(\frac{3\pi}{2}\right) - \sin(2\pi)\right)\left(-\sin\frac{\pi}{2} - \sin(2\pi)\right) - \left(-\sin(2\pi)\right)^2 \not> 0$

$y = x = \pi/3 > 0$

$\cancel{(\pi/2, 3\pi/2)}$ $(\pi/3, \pi/3)$

$\cancel{f(\pi/2, 3\pi/2) = D}$

$f(\pi/3, \pi/3) = \frac{3\sqrt{3}}{2}$ (Max)

Q4. In ΔABC, find max value of $\cos A \cos B \cos C$.

A4. $u = \cos A \cos B \cos C = 0$

$\frac{\partial u}{\partial A} = -\sin A \cos B \cos C = 0$

$\frac{\partial u}{\partial B} = \cos A(-\sin B)\cos C = 0$

$\frac{\partial u}{\partial C} = \cos A \cos B(-\sin C) = 0$

$\angle A + \angle B + \angle C = 180^\circ \Rightarrow \angle A + \angle B + \angle C - 180^\circ = 0 = \alpha$

$\varphi = u + \lambda \alpha$

$\frac{\partial \varphi}{\partial A} = -\sin A \cos B \cos C + \lambda = 0$

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Q. $u = \frac{x-y}{x+z}$ $\quad v = \frac{x+z}{y+z}$, find relation

$$\frac{\partial(u,v)}{\partial(x,y)} = 0 = \frac{\partial(u,v)}{\partial(y,z)} = 0 = \frac{\partial(u,v)}{\partial(z,x)}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{(x+z)-(x-y)}{(x+z)^2} & \frac{-1}{x+z} \\ \frac{1}{y+z} & \frac{-(x+z)}{(y+z)^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{z+y}{(x+z)^2} & \frac{-1}{x+z} \\ \frac{1}{y+z} & \frac{-(x+z)}{(y+z)^2} \end{vmatrix}$$

$$= \frac{z+y}{(x+z)^2} \cdot \frac{(-)(x+z)}{(y+z)^2} = \frac{-1}{(x+z)(y+z)} + \frac{1}{(x+z)(y+z)} = 0$$

$$\frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{-1}{x+z} & \frac{-(x-y)}{(x+z)^2} \\ \frac{-(x+z)}{(y+z)^2} & \frac{(y+z)-(x+z)}{(y+z)^2} \end{vmatrix}$$

$$= \left(\frac{-1}{x+z} \cdot \frac{y-x}{(y+z)^2}\right) - \left(\frac{(x-y)}{(x+z)^2} \cdot \frac{(x+z)}{(y+z)^2}\right)$$

$$= 0$$

||ly $\frac{\partial(u,v)}{\partial(x,z)} = 0$

Hence functionally dependent.

$$\boxed{v = \frac{1}{1-u} \text{ Ans.}}$$

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Unit-II Differential Equations

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$$

⇒ Order & Degree of diff. Eq^n :-

(a) $\frac{dy}{dx} + 2\frac{d^2y}{dx^2} = 3xy$ Order = 2

(b) $\frac{dy}{dx} + \sqrt{1+\frac{d^4y}{dx^4}} = 2$ Order = 4
Degree = 1 (power of order)

$$\left(1+\frac{d^4y}{dx^4}\right)^{1/2} = 2-\frac{dy}{dx}$$

$$1+\frac{d^4y}{dx^4} = \left(2-\frac{dy}{dx}\right)^2$$

(c) $\left(\frac{dy}{dx}\right)^2 + y = 2x$ Order = 1
Degree = 2

GENERAL FORMS

(1) 1st order 1st degree :-

$$F\left(x, y, \frac{dy}{dx}\right) = 0$$

Order decides no. of arbitary constt in sol^n.
Here total no. of arbitary constts will be 1 in the sol^n.

(2) 1st Order n^{th} degree :-

$$G\left(x, y, \frac{dy}{dx}, \left(\frac{dy}{dx}\right)^2, \left(\frac{dy}{dx}\right)^3, \ldots, \left(\frac{dy}{dx}\right)^n\right) = 0$$

arbitary constt = 1

(3) n^{th} order 1st degree :-

$$H\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$$

n arbitary constt will appear in soln.

Obtain a differential equation from solution :-

Q1. $Ax^2 + By^2 = 1$

$$A2x + B2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2Ax}{2By} = \frac{-Ax}{By} \quad -(i)$$

$$\cancel{2A + 2B\frac{d^2y}{dx^2} = 0}$$

$$Ax + By\frac{dy}{dx} = 0$$

$$A + B\left[\left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right)^2\right] = 0$$

$$\frac{-A}{B} = \left(\frac{dy}{dx}\right)^2 + y\left(\frac{d^2y}{dx^2}\right) \quad -(ii)$$

$$(i) = \frac{ydy}{xdx} = \frac{-A}{B}$$

$$\frac{ydy}{xdx} = \left(\frac{dy}{dx}\right)^2 + y\left(\frac{d^2y}{dx^2}\right)$$

$$\left(\frac{dy}{dx}\right)^2 + y\left(\frac{d^2y}{dx^2}\right) - \frac{y}{x}\frac{dy}{dx} = 0$$

Q2. Find diff eqn of all circles of radius 'A'

General eqn of circle with radius A :-

$$(x-h)^2 + (y-k)^2 = A^2$$

arbitary constt (h, k)

$$2(x-h) + 2(y-k)\frac{dy}{dx} = 0$$

$$(x-h) + (y-k)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(x-h)}{(y-k)}$$

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$$(1)+\left(\frac{dy}{dx}\right)^2+(y-k)\frac{d^2y}{dx^2}=0$$

$$1+\left(\frac{dy}{dx}\right)^2+(y-k)\frac{d^2y}{dx^2}=0$$

$$(y-k)=\frac{\left(\frac{dy}{dx}\right)^2-1}{\left(\frac{d^2y}{dx^2}\right)}=\frac{-(1+(y')^2)}{y''}$$

$$(x-h)+\left[-\left(\frac{1+(y')^2}{y''}\right)\right]\frac{dy}{dx}=0.$$

$$x-h=\left(\frac{1+(y')^2}{y''}\right)y'$$

$$\left[\left(\frac{1+(y')^2}{y''}\right)y'\right]^2+\left[\frac{-(1+y')^2}{y''}\right]^2=A^2$$

$$\left(\frac{1+(y')^2}{y''}\right)^2\left((y')^2+1\right)=A^2$$

$F\left(x,y,\frac{dy}{dx}\right)=0$ FIRST ORDER FIRST DEGREE.

(1.) Variable Separable :-

Q1. $\frac{dy}{dx}=e^{x+y}+e^{y}x^2$

$$\frac{dy}{dx}=e^{y}(e^{x}+x^2)$$

$$\int e^{-y}\,dy=\int dx\,(e^{x}+x^2)$$

$$-e^{-y}=e^{x}+\frac{x^3}{3}+c \Rightarrow e^{x}+\frac{x^3}{3}+e^{-y}+c=0$$

Q2. $\cos(x+y)\,dy = dx$

$$\cos(x+y) = \frac{dx}{dy}$$

$$\frac{dy}{dx} = \sec(x+y)$$

put $z = x+y$, then $\frac{dz}{dx} = 1 + \frac{dy}{dx}$

$$\frac{dz}{dx} - 1 = \sec z$$

$$\frac{dz}{dx} = \sec z + 1$$

$$\int \frac{1}{1+\sec z}\,dz = \int dx$$

$$\int \frac{1}{1+\frac{1}{\cos z}} = \int \frac{\cos z}{1+\cos z}\,dz = dx$$

$$1 - \frac{1}{1+\cos z}\,dz = dx$$

$$\int \left(1 - \frac{1}{2\cos^2 z/2}\right) dz = \int dx$$

$$\int \left(1 - \frac{1}{2}\sec^2 z/2\right) dz = \int dx$$

$$z - \underline{\quad}\tan z/2 = x + c$$

$$x+y - \underline{\quad}\tan\left(\frac{x+y}{2}\right) = x + c$$

$$y - \underline{\quad}\tan\left(\frac{x+y}{2}\right) = c$$

2. Homogeneous form:-

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$ have same degree

Put $y = vx$

Q1. $x^2 dy + y(x+y)\,dx = 0$

$$\frac{dy}{dx} = \frac{-xy - y^2}{x^2}$$

Put $y = vx$

then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{-x(vx) - (vx)^2}{x^2} = -v - v^2$$

$$v + x\frac{dv}{dx} = -v - v^2$$

$$x\frac{dv}{dx} = -2v - v^2$$

$$\int \frac{-dv}{2v + v^2} = \int \frac{dx}{x}$$

$$-\int \frac{dv}{v(2+v)} = \int \frac{dx}{x}$$

$$\frac{dv}{v(2+v)} = \frac{A}{v} + \frac{B}{(2+v)}$$

$dv = A(2+v) + B(v) = 1$

put $v = 0$

$2A = 1$ $\boxed{A = 1/2}$

put $v = -2$ $\boxed{B = -1/2}$

$$\frac{dv}{v(2+v)} = \frac{1}{2v} + \left(\frac{-1}{2}\right)\frac{1}{(2+v)}$$

$$-\int \frac{dv}{2v} + \int \frac{dv}{2(v+2)} = \int \frac{dx}{x}$$

$$-\frac{1}{2}\ln v + \frac{1}{2}\ln(v+2) = x\ln x + \ln c$$

$$\ln(v+2) - \ln(v) = \ln(xc)^2$$

$$\ln\left(\frac{v+2}{v}\right) = \ln(xc)^2$$

$$\frac{v+2}{v} = x^2c^2$$

$$= \frac{\frac{y}{x}+2}{y/x} = x^2c^2$$

$$= \frac{\frac{y+2x}{x}}{\frac{y}{x}} = x^2c^2$$

$$\Rightarrow \boxed{\frac{y+2x}{y} = x^2c^2}$$

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EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM:-

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$$

Case 1:- $\frac{a}{A} = \frac{b}{B} = \frac{1}{m}$

$\Rightarrow A = am$, $B = bm$

$$\frac{dy}{dx} = \frac{ax+by+c}{amx+bmy+C}$$

$$\frac{dy}{dx} = \frac{ax+by+c}{m[ax+by]+C}$$

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$z = ax + by$

$\frac{dz}{dx} = a + b\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \left(\frac{dz}{dx} - a\right)\frac{1}{b}$

$\frac{1}{b}\left(\frac{dz}{dx} - a\right) = \frac{z + c}{mz + C}$

Case 2:- $\frac{a}{A} \neq \frac{b}{B}$

Put $x = X + h$; $y = Y + k$

choose h & k s.t $ah + bk + c = 0$

$Ah + Bk + C = 0$

$\frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{A(X+h) + B(Y+k) + C}$

$= \frac{aX + bY + (ah + bk + c)}{AX + BY + (Ah + Bk + C)}$

$= \frac{aX + bY}{AX + BY}$; Put $Y = VX$

Q1 $\frac{dy}{dx} = \frac{x - 2y + 5}{2x + y - 1}$

$a = 1$, $b = -2$, $c = 5$, $A = 2$, $B = 1$, $C = -1$

$\frac{a}{A} = \frac{1}{2}$; $\frac{b}{B} = \frac{-2}{1}$ $\quad \frac{a}{A} \neq \frac{b}{B}$

Subst. $x = X + h$ $\quad y = Y + k$

$\frac{dY}{dX} = \frac{(X+h) - 2(Y+k) + 5}{2(X+h) + (Y+k) - 1}$

$= \frac{X - 2Y + (h - 2k + 5)}{2X + Y + (2h + k - 1)}$

choose h & k st

$h - 2k + 5 = 0 \Rightarrow h = -3/5$

$2h + k - 1 = 0 \Rightarrow k = 11/5$

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Eq^n reduces to $\frac{dy}{dx} = \frac{x-2y}{2x+y}$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x-2(vx)}{2x+(vx)} = \frac{1-2v}{2+v}$$

$$x\frac{dv}{dx} = \frac{1-2v-2v-v^2}{2+v} = \cancel{\frac{-1-3v}{2+v}} \quad \frac{1-4v-v^2}{2+v}$$

$$\cancel{\int \frac{-(2+v)}{(1+3v)} \frac{dv}{d}} = \int \frac{dx}{x}$$

$$\int \frac{2+v}{1-4v-v^2} dv = \int \frac{dx}{x}$$

$$-\int \frac{2v}{v^2+4v-1} dv = \int \frac{dx}{x}$$

$$v^2 + 2(2v) + 4 - 5$$

$$-\int \frac{2v}{(v+2)^2-5} dv = \int \frac{dx}{x}$$

mthd-1

mthd-2.

$$v^2 + 4v - 1 = z$$

$$(2v+4)dv = dz$$

$$-\frac{1}{2}\int \frac{dz}{z} = \int \frac{dx}{x}$$

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$-\frac{1}{2}\ln(v^2+4v-1) = \ln x + \ln c$

$-\ln(v^2+4v-1) = \ln x^2 + (\ln c)'$

$\ln(v^2+4v-1) + \ln x^2 = (\ln c)'$

$(v^2+4v-1)(x^2) = c'$

$\left(1 - 4\frac{y}{x} - \frac{y^2}{x^2}\right)x^2 = -c' = c''$

$(x^2 - 4xy - y^2) = c''$

$\Rightarrow \left(x+\frac{3}{5}\right)^2 - 4\left(x+\frac{3}{5}\right)\left(y-\frac{1}{5}\right) - \cancel{y}\left(y-\frac{1}{5}\right)^2 = c''$

Q2 $(2x+y+1)\,dx + (4x+2y-1)\,dy = 0$

$\frac{-(2x+y+1)}{(4x+2y-1)} = \frac{dy}{dx}$

$a=-2,\ b=-1,\ c=-1$

$A=4,\ B=2,\ C=-1$

$\frac{a}{A} = \frac{-2}{4} = \frac{-1}{2} \equiv \frac{b}{B} = \frac{-1}{2}$

$\frac{dy}{dx} = \frac{-(2x+y+1)}{2(2x+y)-1}$

Put $z = 2x+y$

$\frac{dz}{dx} = 2 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} - 2$

$\left(\frac{dz}{dx} - 2\right) = \frac{-z-1}{2z-1} \Rightarrow \frac{dz}{dx} = \frac{-z-1}{2z-1} + 2 = \frac{-z-1+4z-2}{2z-1}$

$= \frac{3z-3}{2z-1}$

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$$\frac{dz}{dx} = \frac{3z-3}{2z-1}$$

$$\int \frac{(2z-1)}{(3z-3)}\,dz = \int dx$$

$$\frac{1}{3}\int \frac{2z-1}{z-1}\,dz = \int dx$$

$$\frac{1}{3}\int \frac{z-1}{z-1} + \frac{z}{z-1}\,dz = \int dx$$

$$\frac{1}{3}\int 1 + \left(\frac{z-1+1}{z-1}\right) dz = \int dx$$

$$\frac{1}{3}\int \left(1+1+\frac{1}{z-1}\right) dz = \int dx$$

$$\frac{1}{3}\left[\int 2dz + \int \frac{1}{z-1}\,dz\right] = \int dx$$

$$\frac{1}{3}\left[2z + \ln(z-1)\right] = x + c$$

$$2z + \ln(z-1) = 3x + c'$$

$$2(2x+y) + \ln(2x+y-1) = 3x + c'$$

$$x + 2y + \ln(2x+y-1) = c'$$

Q3. $\frac{dy}{dx} = \frac{2x+y-1}{x+4y+5}$

$a=2,\ b=1,\ c=-1,\ A=1,\ B=4,\ C=5$

$\frac{a}{A} = \frac{2}{1} \neq \frac{b}{B} = \frac{1}{4}$

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Choose h & k such that

$2h + k - 1 = 0$

$h + 4k + 5 = 0$ (×2) $\Rightarrow 2h + 8k + 10 = 0$

$\cancel{2h} + 8k + 10 - \cancel{2h} - k + 1 = 0$

$7k + 11 = 0$

$\boxed{k = -11/7}$

$2h + \left(\frac{-11}{7}\right) - 1 = 0$

$14h - 11 - 7 = 0$

$14h - 18 = 0$

$\boxed{h = 18/14 = 9/7}$

$x = X + h \;;\; y = Y + k$

$$\frac{dy}{dx} = \frac{2(X+h) + (Y+k) - 1}{(X+h) + 4(Y+k) + 5}$$

$$\frac{dy}{dx} = \frac{2X + Y + (2h + k - 1)}{X + 4Y + (h + 4k + 5)} = \frac{2X + Y}{X + 4Y}$$

$Y = VX$

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

$$V + X\frac{dV}{dX} = \frac{2X + VX}{X + 4X V} = \frac{2 + V}{\cancel{X} + 4V}$$

$$X\frac{dV}{dX} = \frac{2 + V}{\cancel{X} + 4V} - V$$

$$X\frac{dV}{dX} = \frac{2 + \cancel{V} - \cancel{V} - 4V^2}{1 + 4V} = \frac{2 - 4V^2}{1 + 4V}$$

$$\int \frac{1 + 4V}{2 - 4V^2}\, dV = \int \frac{dX}{X}$$

$$\frac{1}{2}\int \frac{1 + 4V}{1 - 2V^2}\, dV = \int \frac{dX}{X}$$

$$\frac{1}{2}\int \frac{1+4v}{1-2v^2}\,dv = \int \frac{dx}{x}$$

$$\frac{1}{2}\left[\int \frac{1}{1-2v^2}\,dv + \int \frac{4v\,dv}{1-2v^2}\right] = \int \frac{dx}{x}$$

let $s = 1-2v^2 \Rightarrow ds = -4v\,dv$

$$\frac{1}{2}\left[\int \frac{1}{1-2v^2}\,dv - \int \frac{ds}{s}\right] = \int \frac{dx}{x}$$

$$\frac{1}{2}\left[\underbrace{\int \frac{1}{1-2v^2}\,dv}_{I} - \int \frac{ds}{s}\right] = \int \frac{dx}{x}$$

$$I = \int \frac{1}{1-2v^2} = \frac{1}{(1)^2-(\sqrt{2}v)^2}$$

$$= \frac{1}{2}\ln\left|\frac{1+\sqrt{2}v}{\sqrt{2}v-1}\right|$$

$$\frac{1}{2}\left[\frac{1}{2}\ln\left|\frac{1+\sqrt{2}v}{\sqrt{2}v-1}\right| - \ln s\right] = \ln x + \ln c$$

$$\frac{1}{4}\ln\left|\frac{1+\sqrt{2}\,y/x}{\sqrt{2}\frac{y}{x}-1}\right| - \frac{1}{2}\ln\left(1-2\left(\frac{y}{x}\right)^2\right) = \ln xc$$

$$\ln\left|\frac{x+\sqrt{2}y}{\sqrt{2}y-x}\right|^{1/4} - \ln\left(\frac{1-2y^2}{x^2}\right)^{1/2} = \ln xc$$

$h = 9/7$
$k = -11/7$

$$\left(\frac{(x-h)+\sqrt{2}(y-k)}{\sqrt{2}(y-k)-(x-h)}\right)^{1/4} \times \left(\frac{(x-h)^2}{1-2(y-k)^2}\right)^{1/2} = (x-h)c$$

$$\left(\frac{(x-9/7)+\sqrt{2}(y+11/7)}{\sqrt{2}(y+11/7)-(x-9/7)}\right)^{1/4} \times \left(\frac{(x-9/7)^2}{1-2(y+11/7)^2}\right)^{1/2} = (x-9/7)c$$

Ans.

(C) Linear Equations

(1) Leibnitz Form :-

(a) $\frac{dy}{dx} + Py = Q$; where P & Q are fts of x alone

$I.F = e^{\int P dx}$

Sol^n is

$y(I.F) = \int Q(I.F)\,dx + c$

(b) $\frac{dx}{dy} + R\cancel{y}x = S$ where R & S are fts of y only

$I.F = e^{\int R dy}$

Sol^n is :-

$x(I.F) = \int S(I.F)\,dy + c$

Q1. $\frac{dy}{dx} + 2y\tan x = \sin x$; find the sol^n when $y = 0$, for $x = \pi/3$

A1. $P = 2\tan x$, $Q = \sin x$

$x = \sec y$

$\frac{dx}{dy} = \sec y \tan y$

$I.F = e^{\int 2\tan x\, dx} = \sec^2 x$

Sol^n is

$$y\sec^2 x = \int \sin x \cdot \sec^2 x\, dx + c$$

$$= \int \sin x \cdot \frac{1}{\cos^2 x}\, dx + c$$

$$= \int \tan x \cdot \sec x\, dx + c$$

$$y\sec^2 x = \sec x + c$$

$x = \frac{\pi}{3}$, $y = 0$

$$0 = \sec\left(\frac{\pi}{3}\right) + c$$

$\boxed{c = -2}$ $\Rightarrow$ $\boxed{y\sec^2 x = \sec x - 2}$ Ans.

ILATE

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Q2. $(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$

$$\frac{dx}{dy} + \frac{1}{1+y^2}\cdot x = \frac{\tan^{-1}y}{1+y^2}$$

$$I.F = e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Sol^n is

$$x.e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}\cdot e^{\tan^{-1}y}\,dy + c$$

$$t = \tan^{-1}y$$

$$dt = \frac{1}{1+y^2}dy$$

$$x\cdot e^{\tan^{-1}y} = \int t\cdot e^t\,dt + c$$

$$\int \overset{1}{t}\cdot \overset{2}{e^t} = t.e^t - \int \cancel{\frac{t^2}{2}}\cdot e^t = (t-1)e^t$$

$$\cancel{= t.e^t - \frac{1}{2}\int t^2.e^t}$$

$$x e^{\tan^{-1}y} = (\tan^{-1}y - 1)e^{\tan^{-1}y} + c$$

$$x = (\tan^{-1}y - 1) + c e^{-\tan^{-1}y}$$

$$x = (\tan^{-1}y - 1) + ce^{-\tan^{-1}y}$$ **Ans.**

2) Bernoulli's Form :-

$$\frac{dy}{dx} + Py = Qy^n$$

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Q1. $x\frac{dy}{dx} + y = x^3y^6$

$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y^5} = x^2$

Put $t = \frac{1}{y^5}$ $\quad dt = \frac{-5}{y^6}\frac{dy}{dx}$

$\frac{dy}{dx} \Rightarrow \frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$

$-\frac{1}{5}\frac{dt}{dx} + \frac{1}{x}t = x^2$

$\frac{dt}{dx} - \frac{5}{x}t = -5x^2$

$I.F = e^{\int -5/x\,dx} = 1/x^5$

Soln is

$t\left(\frac{1}{x^5}\right) = \int -5x^2\left(\frac{1}{x^5}\right)dx + c$

$\frac{t}{x^5} = -\int \frac{5}{x^3}dx + c$

$\frac{1}{y^5}\cdot\frac{1}{x^5} = -\frac{5}{2}\frac{1}{x^2} + c$ Ans.

Q2 $xy(1+xy^2)\frac{dy}{dx} = 1$

$\frac{dx}{dy} + Rx = Sx^n$

$xy(1+xy^2)\frac{dy}{dx} = 1$

$\frac{dx}{dy} \cancel{+} - xy = x^2y^3$

$xy + x^2y^3 = \frac{dx}{dy}$

$\frac{dx}{dy} - x^2y^3 = xy \Rightarrow \frac{dx}{dy} - xy(xy^2+1) = 0$

$$\frac{dx}{dy} - xy = x^2y^3$$

$$\frac{dx}{dy}\frac{1}{x^2} - y\frac{1}{x} = y^3$$

Subsit. $t = \frac{-1}{x}$

$$\frac{dt}{dy} = +\frac{1}{x^2}\frac{dx}{dy}$$

$$\frac{dt}{dy} + yt = y^3$$

$$I.F = e^{\int y\,dy} = e^{y^2/2}$$

Sol^n :-

$$te^{y^2/2} = \int\left[y^3 \cdot e^{y^2/2}dy\right] + c$$

$e^{f(x)}(f(x)$

$\frac{y^2}{2} +$

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⇒ Exact differential Equation :-

$$Mdx + Ndy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{exact}.$$

$$\underbrace{\int M dx}_{\text{keep } y \text{ const.}} + \int \underbrace{(N \sim x)}_{\text{does not involve } x} dy = c$$

Q1. $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

$M = 5x^4 + 3x^2y^2 - 2xy^3$

$N = 2x^3y - 3x^2y^2 - 5y^4$

$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$

$\frac{\partial N}{\partial x} = 6x^2y - 6xy^2$

$\int (5x^4 + 3x^2y^2 - 2xy^3)dx + \int -5y^4 dy = c$

$(x^5 + x^3y^2 - x^2y^3) - y^5 = c$

Q2 $\left[\cos x \log_e(2y-8) + \frac{1}{x}\right] dx + \frac{\sin x}{y-4} dy = 0$

$\frac{\partial M}{\partial y} = \frac{\cos x}{2y-8}(2)$ $\quad M = \cos x \log_e(2y-8) + \frac{1}{x}$

$\frac{\partial N}{\partial x} = \frac{\cos x}{y-4}$ $\quad N = \frac{\sin x}{y-4}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact eq^n.

$$\int \left(\cos x \log_e(2y-8) + \frac{1}{x}\right) dx + \int 0\, dy = c$$

$$\sin x \log(2y-8) + \ln x + 0 = c$$

FORMULAS OF INTEGRAL :-

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

(b) $\int dx = x + C$

(c) $\int a^x dx = \frac{a^x}{\log a} + C$

(d) $\int e^x dx = e^x + C$

(e) $\int \frac{1}{x} dx = \log_e |x| + C$

(f) $\int \sec x \tan x \, dx = \sec x + C$

(g) $\int \tan x \, dx = \ln |\sec x| + C$

(h) $\int \cot x \, dx = \ln |\sin x| + C$

(i) $\int \sec x \, dx = \ln |\sec x + \tan x| + C$

(j) $\int \operatorname{cosec} x \, dx = \ln |\operatorname{cosec} x - \cot x| + C$

(k) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

(l) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

(m) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left(x + \sqrt{x^2 \pm a^2}\right) + C$

(n) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left|\frac{x-a}{x+a}\right| + C$

(o) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left|\frac{a+x}{x-a}\right| + C$

Partial fraction

$$\frac{(a)\,dx}{(b)(c)} = \frac{A}{(b)} + \frac{B}{(c)}$$

By Parts

ILATE :- first function

$$\int (\mathrm{I} \cdot \mathrm{II})\, dx = \mathrm{I}\left(\int \mathrm{II}\, dx\right) - \int \left(\mathrm{I}' \cdot \int \mathrm{II}\, dx\right) dx$$

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(a) $\int x e^x dx = xe^x - e^x + c$

(b) $\int \ln x \, dx = x \ln x - x + c$

(c) $\int \left(f(x) + x f'(x) \right) dx = x f(x) + c$

(d) $\int e^x \left(f(x) + f'(x) \right) dx = e^x f(x) + c$

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Exact Differential Eq^n :-

$$M dx + N dy = 0$$

exact $\Leftrightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Sol^n:-

$$\underset{\text{Keeping } y \text{ const}}{\int M dx} + \int (N \sim x) dy = c$$

Equations Reducible to Exact Diff equation :-

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Leftrightarrow$ not exact

(1) If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \frac{1}{N}$ is a ft. of x alone say $f(x)$ then $I.F = e^{\int f(x) dx}$

(2) If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \frac{1}{M}$ is a ft. of y alone say $g(x)$ then $I.F = e^{-\int g(y) dy}$

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(3) If $M = y f_1(xy)$ & $N = x f_2(xy)$

then $I.F = \frac{1}{Mx - Ny}$; $Mx - Ny \neq 0$.

(4) If $x^a y^b (\alpha_1 y dx + \alpha_2 x dy) + x^c y^d (m_1 y dx + m_2 x dy) = 0$

$I.F = x^{\alpha} y^{\beta}$

(5) If $Mdx + Ndy = 0$ such that $Mx + Ny \neq 0$
then $IF = \frac{1}{Mx + Ny}$

Multiply eq^n with IF & solve new equation.

Q1. $(x^2 + y^2 + 1) dx - 2xy\, dy = 0$

$M = x^2 + y^2 + 1$

$N = -2xy$

$\frac{\partial M}{\partial y} = 2y \neq \frac{\partial N}{\partial x} = -2y$

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{1}{N} = \frac{2y - (-2y)}{2xy} = \frac{4y}{-2xy} = \frac{-2}{x} = f(x)$$

$$I.F = \int e^{-2/x\, dx} = \frac{1}{x^2}$$

$\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx - \frac{2y}{x} dy = 0$ is the new exact diff eq^n

$$\frac{\partial M}{\partial y} = \frac{2y}{x^2} = \frac{\partial N}{\partial x} = \frac{+2y}{x^2}$$

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solt is

$$\int\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx+\int 0\,dy=0$$

$$x-\frac{y^2}{x^2}-\frac{1}{x}=c \quad \underline{\text{Ans.}}$$

<u>Q2.</u> $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$

$M=y^4+2y \qquad N=xy^3+2y^4-4x$

$$\frac{\partial M}{\partial y}=4y^3+2 \neq \frac{\partial N}{\partial x}=y^3-4$$

NOT EXACT

$$\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\frac{1}{M}=\frac{3y^3+6}{y(y^3+2)}=\frac{3}{y}=g(y)$$

$$IF=e^{-\int 3/y\,dy}=1/y^3$$

new eqⁿ is $\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0$

$$\frac{\partial M}{\partial y}=1-\frac{4}{y^3} \qquad \frac{\partial N}{\partial x}=1-\frac{4}{y^3}$$

$$\int M dx+\int(N-x)dy$$

$$\int\left(y+\frac{2}{y^2}\right)dx+\int 2y\,dy$$

$xy+2x/y^2+y^2=c$

$$xy+\frac{2x}{y^2}+y^2=c$$

y^{-2}

$-2y^{-3}$

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Q3. $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

$\frac{\partial M}{\partial y} = x^2 - 4yx \neq \frac{\partial N}{\partial x} = -3x^2 + 6xy$

$Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2$
$= x^2y^2 \neq 0$

$IF = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

New eqn :-

$\left(\frac{x^2y}{x^2y^2} - \frac{2xy^2}{x^2y^2}\right)dx - \left(\frac{x^3}{x^2y^2} - \frac{3x^2y}{x^2y^2}\right)dy = 0$

$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$

$\frac{\partial M}{\partial y} = \frac{-1}{y^2} \cancel{-2y} = \frac{\partial N}{\partial x} = \frac{-1}{y^2}$

$\int M dx + \int (N \sim x) dy = 0$

$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int\left(\frac{3}{y}\right)dy = c$

$\frac{x}{y} - 2\ln x + 3\ln y = c$

$\frac{x}{y} + \ln\left|\frac{y^3}{x^2}\right| = c$

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Q4. $\cancel{3(x}\ (3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x \qquad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$Mx = 3x^3y^4 + 2x^2y$$
$$Ny = 2x^3y^4 - x^2y$$

$$Mx + Ny = 3x^3y^4 + 2x^2y + 2x^3y^4 - x^2y$$
$$= 5x^3y^4 + x^2y$$

$$IF = \frac{1}{5x^3y^4 + x^2y} = \frac{1}{x^2y(5xy^3+1)}$$

$$\left(\frac{3x^2y^{\cancel{4}3}}{x^2y(5xy^3+1)} + \frac{2xy}{x^2y(5xy^3+1)}\right)dx + \left(\frac{2x^{\cancel{3}}y^{\cancel{3}2}}{x^2y(5xy^3+1)} - \frac{x^2}{x^2y(5xy^3+1)}\right)dy$$

$$\left(\frac{3y^3}{(5xy^3+1)} + \frac{2}{x(5xy^3+1)}\right)dx + \left(\frac{2xy^2}{(5xy^3+1)} - \frac{1}{y(5xy^3+1)}\right)dy = 0$$

Since, Integration is difficult we will try other method.

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{6x^2y^3 + 4x}{3x^2y^4 + 2xy} = \frac{2}{y}$$

$$IF = e^{\int -2/y\, dy} = -1/y^2$$

New eqn is $\left(-3x^2y^2 - \frac{2x}{y}\right)dx + \left(-2x^3y + \frac{x^2}{y^2}\right)dy = 0$

*TBC

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Q5. $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$

$M = y\,f_1(xy) \quad N = x\,f_2(xy)$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{xy[\cancel{xy} + 2x^2y^2 - \cancel{xy} + x^2y^2]}$$
$$= \frac{1}{3x^3y^3}$$

New eqn :- $\left(\frac{xy^2}{3x^3y^3} + \frac{2x^3y^2}{3x^3y^3}\right)dx + \left(\frac{x^2y}{3x^3y^3} - \frac{x^3y^2}{3x^3y^3}\right)dy$

$$0 = \left(\frac{1}{3x^2y} + \frac{2}{3}\cdot\frac{1}{xy}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3yx}\right)dy$$

$$\frac{\partial M}{\partial y} = \frac{-1}{3x^2y^2} - \frac{2}{3xy^2}$$

$\frac{1}{x}$

$$\frac{\partial N}{\partial x} = \frac{-1}{3x^2y^2} - \frac{1}{3yx^2} \quad (\Rightarrow \text{Some error})$$

x^{-1}

$-1x^{-2}$

Original eqn :- $\left[\frac{1}{3x^2y} + \frac{2}{3x}\right]dx + \left[\frac{1}{3xy^2} - \frac{1}{3y}\right]dy = 0$

$$\frac{\partial M}{\partial y} = \frac{-1}{3x^2y^2} = \frac{\partial N}{\partial y} = \frac{-1}{3x^2y^2}$$

$$\int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int\left(\frac{-1}{3y}\right)dy = c$$

$$\frac{-1}{9x^3y} + \frac{2}{3}\ln|x| + \left(\frac{-1}{3}\right)\ln|y| = c$$

Ans

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Date:- 16-12-22.

Q1. $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0$

$(4xy + 3y^4)\,dx + (2x^2 + 5xy^3)\,dy = 0$

$\frac{\partial M}{\partial y} = 4x + 12y^3 \neq \frac{\partial N}{\partial x} = 4x + 15y^3$

choose I.F $= x^\alpha y^\beta$

New Eq^n is

$(4x^{\alpha+1}y^{\beta+1} + 3x^{\alpha}y^{\beta+4})\,dx + (2x^{\alpha+2}y^{\beta} + 5x^{\alpha+1}y^{\beta+3})\,dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ should be true.

$4x^{\alpha+1}(\beta+1)y^{\beta} + 3(\beta+4)x^{\alpha}y^{\beta+3} = 2(\alpha+2)x^{\alpha+1}y^{\beta} + 5(\alpha+1)x^{\alpha}y^{\beta+3}$

comparing coefficients

$4(\beta+1) = 2(\alpha+2)$ — (i)

& $3(\beta+4) = 5(\alpha+1)$ — (ii)

Solving (i) & (ii), $\boxed{\alpha=2}$; $\boxed{\beta=1}$

Exact diff eqn is

$(4x^3y^2 + 3x^2y^5)\,dx + (2x^4y + 5x^3y^4)\,dy = 0$

$\frac{\partial M}{\partial y} = 8x^3y + 15x^2y^4 = \frac{\partial N}{\partial x} = 8x^3y + 15x^2y^4$

$\int(4x^3y^2 + 3x^2y^5)\,dx + \int 0\,dy = 0$

$x^4y^2 + x^3y^5 = c$ Ans.

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Q. $(xy \sin xy + \cos xy) y\,dx + (xy \sin xy - \cos xy) x\,dy = 0$

$(xy^2 \sin xy + y\cos xy)dx + (x^2 y \sin xy - x\cos xy)dy = 0$

$$\frac{\partial M}{\partial y} = 2xy \sin xy + xy^2 \cos xy \cdot x + \cos xy - y \sin xy \cdot x$$

$$= x^2y^2 \cos xy + 2xy \sin xy - xy \sin xy + \cos xy$$

$$= \cos xy (x^2y^2+1) + \sin xy (2xy - xy)$$

$$\frac{\partial N}{\partial x} = 2xy \sin xy + x^2 y \cos xy \cdot y - \cos xy + x \sin xy \cdot y$$

$$= x^2y^2 \cos xy + 2xy \sin xy - \cos xy + xy \sin xy$$

$$= \cos xy (x^2y^2 - 1) + 3xy \sin xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$Mx - Ny = x^2y^2 \sin xy + xy \cos xy - x^2y^2 \sin xy + xy \cos xy$$

$$= 2xy \cos xy$$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{2xy \cos xy}$$

New Eqn is,

$$\left[\frac{y}{2} \tan(xy) + \frac{1}{2x}\right] dx + \left[\frac{x}{2} \tan xy - \frac{1}{2y}\right] dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{1}{2}\left[\tan xy + y \sec^2(xy) \cdot x\right]$$

$$\frac{\partial N}{\partial x} = \frac{1}{2}\left[\tan(xy) + x \sec^2 xy \cdot y\right]$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int\left(\frac{y}{2}\tan(xy)+\frac{1}{2}x\right)dx+\int\left(\frac{-1}{2y}\right)dy=c$$

$$\frac{y}{2}\frac{\log|\sec xy|}{y}+\frac{1}{2}\ln|x|-\frac{1}{2}\ln|y|=\ln k$$

$$\frac{x}{y}\sec(xy)=k'$$

Q5. $(2x^2y^2+y)\,dx-(x^3y-3x)\,dy=0$

$$\frac{\partial M}{\partial y}=4x^2y+1 \neq \frac{\partial N}{\partial x}=(3x^2y-3)$$

$$I.F=x^{\alpha}y^{\beta}$$

$$(2x^{\alpha+2}y^{\beta+2}+y^{\beta+1}x^{\alpha})\,dx - (x^{\alpha+3}y^{\beta+1}-3x^{\alpha+1}y^{\beta})\,dy$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\left(2(\beta+2)y^{\beta+1}x^{\alpha+2}+(\beta+1)y^{\beta}x^{\alpha}\right)=\frac{\partial M}{\partial y}$$

$$\frac{\partial N}{\partial x}=-\left[(\alpha+3)x^{\alpha+2}y^{\beta+1}-3(\alpha+1)x^{\alpha}y^{\beta}\right]$$

compairing coefficients

$$2(\beta+2)=-\alpha-3 \Rightarrow 2\beta+4=-\alpha-3$$

& $(\beta+1)=-3(\alpha+1) \Rightarrow \beta+1=+3\alpha+3$

$$2\beta+2=+6\alpha+6$$

$$\cancel{2\beta}+4-\cancel{2\beta}-2=-\alpha-3-6\alpha-6$$

$$\cancel{2=-5\alpha-3}$$

$$7\alpha=-11 \Rightarrow \boxed{\alpha=-11/7}$$

$$\boxed{\beta=\frac{-19}{7}}$$

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$$(2x^{3/7}y^{-5/7}+x^{-11/7}y^{-12/7})dx-(x^{10/7}y^{-12/7}-3x^{-4/7}y^{-11/7})dy=0$$

$$\int(2x^{3/7}y^{-5/7}+x^{-11/7}y^{-12/7})dx-\int 0\,dy=c$$

$$2y^{-5/7}\frac{x^{10/7}}{10/7}+y^{-12/7}\frac{x^{-4/7}}{-4/7}=c$$

$$\frac{7}{5}y^{-5/7}x^{10/7}-\frac{7}{4}y^{-12/7}x^{-4/7}=c$$

Q. $(x^2+y^2+1)dx+x(x-2y)dy=0$

$$\frac{\partial M}{\partial y}=2y \qquad \frac{\partial N}{\partial x}=2x-2y$$

$$Mx+Ny=x^3+y^2x+x+x^2y-2xy^2$$
$$=x^3-xy^2+x^2y$$

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=2y+2y-2x=4y-2x=-2(x-2y)$$

$$I.F=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-\frac{2(x-2y)}{x(x-2y)}=\frac{-2}{x}$$

$$I.F=e^{\int -2/x\,dx}=\frac{1}{x^2}$$

$$\int\left(1+\left(\frac{y}{x}\right)^2+\left(\frac{1}{x^2}\right)\right)dx+\int\left(1-\frac{2y}{x^2}\right)dy=c$$

$$x+y^2\left(\frac{-1}{x}\right)+\left(\frac{-1}{x}\right)+y=c$$

$$x^2-y^2-1+xy=xc.$$

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*** TBC :-**

$$\int\left(-3x^2y^2 - \frac{2x}{y}\right)dx + \int 0\, dy = c$$

$$-x^3y^2 - \frac{x^2}{y} = c$$

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LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS OF N-th ORDER :-

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = X(x)$$

$$\left[D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n\right]y = X(x)$$

$$D = \frac{d}{dx}$$

$$\boxed{f(D)y = X(x)}$$

where $f(D) = D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n$

Step 1 :-

Find Complementary funcn :- (C.F)

$$f(D)y = 0 \quad \text{---(i)}$$

Suppose $y = e^{mx}$ $(m = \text{constt})$ is the soln of (i)

$$De^{mx} = \frac{d}{dx}e^{mx} = me^{mx}$$

$$D^2(e^{mx}) = \frac{d^2}{dx^2}(e^{mx}) = m^2e^{mx}$$

$$D^3(e^{mx}) = \frac{d^3}{dx^3}(e^{mx}) = m^3e^{mx}$$

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$$D^n(e^{mx}) = \frac{d^n}{dx^n}(e^{mx}) = m^n e^{mx}$$

$$[m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_{n-1} m + a_n] e^{mx} = 0$$

$$\boxed{m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_{n-1} m + a_n = 0}$$

→ auxiliary eqn.

Exactly n-Roots for n

say roots are $m_1, m_2, \ldots, m_n$

C.F :-

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}$$

where $C_1, C_2, \ldots, C_n$ are arbitary constts.

Case 1 :-

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

$$[D^2 + a_1 D + a_2] y = 0$$

Auxiliary eqn :- $m^2 + a_1 m + a_2 = 0$

Exactly 2 roots occur m_1 & m_2

Case A :- when m_1 & m_2 are Real & distinct :-

CF = $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

Case B :- $m_1 = m_2 = m$ Real & Equal :-

CF = $y = (C_1 + C_2 x) e^{mx}$

for 3 roots $m_1 = m_2 = m_3 = m$

$$y = (C_1 + C_2 x + C_3 x^2) e^{mx}$$

$m_1 = m_2 = m_3 = m_4 = m$

$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3) e^{mx}$

case C :- $m_1 = a + ib$; $m_2 = a - ib$

complex conjugent Roots

CF :-

$y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$

$= e^{ax} [c_1 e^{ibx} + c_2 e^{-ibx}]$

$y = e^{ax} [c_1(\cos bx + i \sin bx) + c_2(\cos bx - i \sin bx)]$

$= e^{ax} [(c_1 + c_2)\cos bx + i \sin bx (c_1 - c_2)]$

$= e^{ax} [K_1 \cos bx + K_2 \sin bx]$

Step 2 :- To find Particular Integral (P.I.) =

$P.I :- \frac{1}{f(D)} . X(x)$

Step 3 :- Complete Sol^n :-

$\boxed{y = C.F + P.I.}$ * Imp to write

Q1. $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$

Put $D = \frac{d}{dx}$

$[D^3 - 4D^2 + 5D - 2] y = 0$

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Auxilary eqn :-

$m^3 - 4m^2 + 5m - 2 = 0$

Roots :- $m = 1, 2, 1$ (find)

$CF = y = (c_1 + c_2 x)e^{1x} + c_3 e^{2x}$

Q2. $\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = 0$

$[D^2 - 3D + 2]y = 0$

Auxilary eqn :- $m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$m = 1, 2$

$y = c_1 e^x + c_2 e^{2x}$

Q3. $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + y = 0$

$[D^4 + D^2 + 1]y = 0$

$m^4 + m^2 + 1 = 0$ = Auxiliary eqn

$m^4 + m^2 + 1 = \boxed{(x^2 - x + 1)(x^2 + x + 1)}$

$= \underset{A}{(m^2 - m + 1)}\ \underset{B}{(m^2 + m + 1)}$

Solving A = $\frac{1 \pm \sqrt{1 - 4(1)(1)}}{2}$

$= \frac{1 \pm \sqrt{3}i}{2} = -\omega$; $\omega^2 = \frac{1 - \sqrt{3}i}{2}$

Solving B = Roots are $+\omega, +\omega^2$

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$$m = \frac{-1+\sqrt{3}i}{2}, \frac{1+\sqrt{3}i}{2}$$

$$\text{CF / sol}^n \text{ is } = (C_1 + C_2 x)e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right) + e^{x/2}(C_3 + C_4 x)\sin\left(\frac{\sqrt{3}x}{2}\right)$$

Q4. $\frac{d^4y}{dx^4} - \frac{2d^3y}{dx^3} + \frac{5d^2y}{dx^2} - \frac{8dy}{dx} + 4y = 0$

$[D^4 - 2D^3 + 5D^2 - 8D + 4]y = 0$

Auxiliary eq^n = $m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$

$(1 = m) \Rightarrow 1 - 2 + 5 - 8 + 4 = 0$

$(m-1)$ is a root

$(m=-1) \Rightarrow 1 + 2 + 5 + 8 + 4 \neq 0$ not satisfying

$$\begin{array}{r}
m^3 - m^2 + 4m - 4 \\
(m-1) \overline{)\, m^4 - 2m^3 + 5m^2 - 8m + 4} \\
-\cancel{m^4} \mp m^3 \\
\hline
-\cancel{m^3} + 5m^2 \\
\mp \cancel{m^3} \pm m^2 \\
\hline
4\cancel{m^2} - 8m \\
-4m^2 \mp 4m \\
\hline
-4m + 4 \\
\mp \cancel{4m} \pm \cancel{4} \\
\hline
\times
\end{array}$$

$m^3 - m^2 + 4m - 4 = 0$

$(m=1)$ $1 - 1 + 4 - 4 = 0$

$\rightarrow$

$$\begin{array}{r} m^2+4 \\ (m-1)\overline{\big)\, m^3 - m^2 + 4m - 4} \\ \underline{m^3 - m^2} \\ 4m-4 \\ \underline{4m-4} \\ \times \end{array}$$

$(m^2+4)(m-1)(m-1)$

$m^2 = -4$

$m = \cancel{\sqrt{2}i} \; \cancel{+\sqrt{2}i} \; +2i \;\&\; -2i$

Roots :- $1, 1, +2i, -2i$

$CF = y = (C_1 + C_2 x) e^{1x} + e^{0x} [C_3 \cos 2x + C_4 \sin 2x]$

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Particular Integral :-

$$P.(I). = \frac{1}{f(D)} X(x) \quad ; D = \frac{d}{dx}$$

$$\frac{1}{D} X(x) = \int X(x)\,dx$$

(a) PI =

$$\frac{1}{D-a} X(x) = e^{ax} \int e^{-ax} X(x)\,dx$$

(b) PI =

$$\frac{1}{D+a} X(x) = e^{-ax} \int e^{ax} X(x)\,dx$$

Q. $(D^2+a^2)y = \tan ax$

C.F = $(D^2+a^2)y = 0$

Auxiliary eqn :- $m^2 + a^2 = 0$

$m = \pm ia$

roots = $\pm ia$

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$CF \equiv y = (c_1 \cos ax + c_2 \sin ax) e^{0x}$

$PI = \frac{1}{D^2+a^2} \tan ax$

$= \frac{1}{(D+ia)(D-ia)} \tan ax$

$\frac{1}{(D+ia)(D-ia)} = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]$ (using Partial fraction)

$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \tan ax$

$= \frac{1}{2ia}\left[\frac{1}{D-ia} \tan ax - \frac{1}{D+ia} \tan ax\right]$

$\frac{1}{D-ia} \tan ax = e^{iax} \int e^{-iax} \tan ax \, dx$

$= e^{iax} \int [\cos ax - i \sin ax] \frac{\sin ax}{\cos ax} dx$

$= e^{iax} \int \left(\sin ax - \frac{i \sin^2 ax}{\cos ax}\right) dx$

$= e^{iax} \int \left(\sin ax - i\left(\frac{1}{\cos ax} - \frac{\cos^2 ax}{\cos ax}\right)\right) dx$

$= e^{iax} \int (\sin ax - i \sec ax + i \cos ax) \, dx$

$= e^{iax}\left[\frac{-\cos ax}{a} - \frac{i}{a}\log(\sec ax + \tan ax) + \frac{i \sin ax}{a}\right]$

$= -\frac{e^{iax}}{a}\left[(\cos ax - i \sin ax) + i \log(\sec ax + \tan ax)\right]$

$\cos ax + i \sin ax$.

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$$= \frac{-e^{iax}}{a}\left[e^{iax} + i\log(\sec ax + \tan ax)\right]$$

$$\frac{1}{D+ia}\tan ax = e^{-iax}\int e^{iax}\tan ax$$

$$= e^{-iax}\int [\cos ax + i\sin ax]\frac{\sin ax}{\cos ax}\,dx$$

$$= e^{-iax}\int \sin ax + \frac{i\sin^2 ax}{\cos ax}$$

$$= \frac{-e^{-iax}}{a}\left[e^{iax} \cancel{+} i\log(\sec ax + \tan ax)\right]$$

$$PI = \frac{1}{2ia}\left[\frac{-e^{iax}}{a}\left(e^{iax} + i\log(\sec ax + \tan ax)\right) + \frac{e^{-iax}}{a}\left(e^{iax} \cancel{+} i\log(\sec ax + \tan ax)\right)\right.$$

$$= \frac{1}{\cancel{2}ia}\left(-\frac{\cancel{2}e^{iax}}{a}i\log(\sec ax + \tan ax)\right)$$

$$= -\frac{e^{iax}}{a^2}\log(\sec ax$$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$
$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

Ans:- $-\frac{1}{a^2}\cos ax\log\left(\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right)$; complete solⁿ = ?

FORMULAS :-

(c) $PI = \frac{1}{f(D)}e^{ax} = \begin{cases} \frac{1}{f(a)}e^{ax} & ; f(a) \neq 0 \\ \left.\frac{x \cdot e^{ax}}{f'(D)}\right|_{D=a} & ; f(a) = 0 \end{cases}$

(d) $\frac{1}{(D-a)^r\phi(D)}e^{ax} = \frac{x^r e^{ax}}{r!\,\phi(a)}$; $\phi(a) \neq 0$

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Q1. $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = e^x$

CF :- $[D^4 - 2D^3 + 5D^2 - 8D + 4]y = 0$

Auxiliary funcⁿ :-

$m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$

$m = 1, 1, +2i, -2i$

roots $= 1, 1, +2i, -2i$

$CF = (C_1 + C_2x)e^x + C_3\cos 2x + C_4\sin 2x$

$PI = \frac{1}{D^4 - 2D^3 + 5D^2 - 8D + 4} e^x$ denominator $= 0$ on $D = 1$

$= \frac{x \cdot e^x}{4D^3 - 6D^2 + 10D - 8}$ $D = 1$, denom $= 0$

$= \frac{x^2 e^x}{12D^2 - 12D + 10}$ $D = 1$, den $= 10$

$= \frac{x^2 e^x}{10}$

$y = (C_1 + C_2x)e^x + C_3\cos 2x + C_4\sin 2x + \frac{x^2 e^x}{10}$

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Q3. $\dfrac{d^3y}{dx^3} - y = (e^x+1)^2$

$[D^3 - 1]\, y = 0$ CF

Auxiliary eqn :- $m^3 - 1 = 0$ $\quad \omega = \dfrac{-1+\sqrt{3}i}{2}$

roots $= 1, \omega, \omega^2$

$y = \cancel{c_1e^x + c_2e^{\omega x} + c_3e^{\omega^2 x}}\; c_1e^x + e^{-x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\sqrt{3}/2\, x\right]$

$P.I = \dfrac{1}{D^3-1}(e^x+1)^2$

$= \dfrac{1}{D^3-1}(e^{2x}+1+2e^x)$

$= \underbrace{\dfrac{1}{D^3-1}(e^{2x})}_{A} + \underbrace{\dfrac{1}{D^3-1}e^{0x}}_{B} + \underbrace{\dfrac{1}{D^3-1}(2e^x)}_{C}$

$A = \dfrac{1}{D^3-1}(e^{2x}) = \dfrac{1}{7}e^{2x}$

$B = \dfrac{1}{D^3-1}e^{0x} = -1e^{0x}$

$C = \dfrac{1}{D^3-1}(2e^x)$

$= \dfrac{2e^x \cdot x}{3D^2} = \dfrac{2xe^x}{3}$

$PI = 1/7\, e^{2x} - 1 + 2xe^x/3$

$y = c_1e^x + e^{-x/2}\left[c_2\cos\sqrt{3}/2\, x + c_3\sin\frac{\sqrt{3}x}{2}\right]$
$+ \dfrac{1}{7}e^{2x} - 1 + \dfrac{2xe^x}{3}$ Ans.

$y = CF + PI$

Q4 $(D^2+6D+9)y = e^{-3x}$

$CF = (D^2+6D+9)y$

Auxiliary eqn = $m^2+6m+9=0$

$m^2+3m+3m+9=0$

$m(m+3)+3(m+3)=0$

root $r = -3, -3$

Date :- 27 Dec, 2022 (Revise)

FORMULAS FOR P.I.

\# $X(x) = \sin ax / \cos ax$

(e) $PI = \frac{1}{f(D^2)} \sin/\cos ax = \frac{1}{f(-a^2)} \sin/\cos ax$

$f(-a^2) \neq 0$ then $\frac{1}{f(-a^2)} \sin/\cos ax$

$f(-a^2) = 0$ then $\frac{x}{f'(D^2)} \sin/\cos ax \Big|_{D^2=-a^2}$

Q. (f) $X(x) = e^{ax} V$

$PI = \frac{1}{f(D)} e^{ax} V = e^{xa} \frac{1}{f(D+a)} \cdot V$ (e^{ax} is constt.)

(g) $X(x) = x^m$ (polynomial)

$PI =$ $(1+\theta)^{-1} = 1-\theta+\theta^2-\theta^3+\theta^4-\theta^5+\cdots$

$(1-\theta)^{-1} = 1+\theta+\theta^2+\theta^3+\theta^4+\theta^5+\cdots$ Expan

$\frac{1}{f(D)} x^m = f(D)^{-1}(x^m)$

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Q. $(D^2 - 2D + 1)y = \cos 3x$

$m^2 - 2m + 1$ CF $= [D^2 - 2D + 1]y = 0$

roots: $m^2 - m - m + 1$

$m(m-1) - 1(m-1)$

$\boxed{m = 1, 1}$

CF $y = (C_1 + C_2 x)e^x$

$$P.I = \frac{1}{D^2 - 2D + 1} \cos 3x$$

$$= \frac{1}{-(3^2) - 2D + 1} \cos 3x$$

$$= \frac{1}{-2D - 8} \cos 3x$$

$$= -\frac{1}{2} \frac{1}{D+4} \cos 3x$$

$$= \frac{-1}{2} \frac{1}{D+4} \times \frac{D-4}{D-4} \cos 3x$$

$$= \frac{-1}{2} \frac{1}{-9-16} [-3\sin 3x - 4\cos 3x]$$

$$= \frac{-1}{2} \cdot \frac{-1}{25} [-3\sin 3x - 4\cos 3x]$$

$$= \frac{1}{50} [-3\sin 3x - 4\cos 3x]$$

$$y = CF + PI = (C_1 + C_2 x)e^x \mp \frac{1}{50} [+3\sin 3x + 4\cos 3x]$$

Ans.

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Q. $\frac{d^4y}{dx^4} - m^4y = \sin mx$

CF :- $[D^4 - m^4]y = 0$

Auxiliary eqn :- $D^4 - m^4 = 0$

$D^4 = m^4$

$D = \pm m, 0, 0$

$y = C_1 e^{mx} + C_2 e^{-mx}$

— x — x —

$CF = (D^2 + m^2)(D^2 - m^2)y = 0$

Roots :- $\pm im ; \pm m$

$y = C_1 e^{mx} + C_2 e^{-mx} + C_3 \cos mx + C_4 \sin mx$

* $P.I = \frac{1}{(D^2+m^2)(D^2-m^2)} \sin mx$

* $= \frac{-1}{2m^2}\left[\frac{1}{D^2+m^2} \sin mx\right]$

* $= \frac{-1}{2m^2}\left[\frac{x}{2D} \sin mx\right]$

* $= \frac{-x}{4m^2}\left(\frac{1}{D} \sin mx\right)$

$= \frac{-x}{4m^2}\left(\frac{1}{D} \sin mx\right) = \frac{x}{4m^2}\left(\frac{\cos mx}{m}\right)$

$= \frac{x \cos mx}{4m^3}$

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$y = CF + PI$

$= c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + \frac{x \cos mx}{4m^3}$

Q3 $(D^3+8)y = x^4+2x+1$

$m^3+8=0$

$m^3 = -8$

$m = \sqrt[3]{-8} = \cancel{\sqrt{8i}}\ \sqrt[3]{8} \times \sqrt[3]{-1}$ $\quad i = \sqrt{-1}$

$m = 2$

$m = -2$

$m-2 \overline{\big) m^3+8}$

$\Downarrow$

$(a^3+b^3) = (a+b)(a^2+b^2-ab)$

$= (D+2)(D^2+4-2D)$ $\quad D^2-2D+4$

$\Rightarrow D = -2$ $\quad a=1 \quad c=4$

$\quad b=-2$

$\frac{+2 \pm \sqrt{4-4(4)}}{2}$

$\frac{+2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}i$

$y = c_1 e^{-2x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}x}{2}\right)$

$PI = \frac{1}{D^3+8}(x^4+2x+1)$

$= \frac{1}{8\left(1+\frac{D^3}{8}\right)}(x^4+2x+1)$

$$= e^{2x}\left[\frac{1}{49D^2-121}(7\cos x-11\sin x)\right]$$

$$= \frac{e^{2x}}{49(-1)^2-121}(7\cos x-11\sin x)$$

$$= \frac{e^{2x}}{49-121}(7\cos x-11\sin x)$$

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$X(x) = xV$

$$PI = \frac{1}{f(D)} xV = x\left[\frac{1}{f(D)} V\right] - \frac{f'(D)}{f(D)} \cdot \left[\frac{1}{f(D)} V\right]$$

Q1. $(D^2 - 2D + 1)y = x \sin x$

$CF = (D^2 - 2D + 1)y = 0$

AE :- $m^2 - 2m + 1 = 0$ roots - 1, 1

$CF = y = (c_1 + c_2 x)e^x$

$$PI = \frac{1}{D^2 - 2D + 1} x \sin x$$

$$= x \cdot \frac{1}{D^2 - 2D + 1} \sin x - \frac{(2D - 2)}{D^2 - 2D + 1} \cdot \left[\frac{1}{D^2 - 2D + 1} \sin x\right]$$

$$= x \cdot \frac{1}{-1 - 2D + 1} \sin x \; \frac{-2D - 2}{D^2 - 2D + 1} \cdot \left[\frac{1}{-1 - 2D + 1} \sin x\right]$$

$$= \frac{-x}{2} \cdot \left(\frac{1}{D} \sin x\right) + \frac{1}{2}\left(\frac{2D - 2}{D^2 - 2D + 1}\right)(\cos x)$$

$$= \frac{-x}{2}(-\cos x) \; \frac{-1}{2} \; \frac{2D - 2}{} \cdot \left(\frac{-1}{2D} \cos x\right)$$

$$= \frac{+x \cos x}{2} + \frac{1}{4}(2D - 2)(\sin x)$$

$$= \frac{x \cos x}{2} + \frac{1}{4}(2\cos x - 2\sin x)$$

$$y = CF + PI = y = (c_1 + c_2 x)e^x + \frac{x \cos x}{2} + \frac{1}{4}(2\cos x - 2\sin x)$$

Q. $(D^2-2D+1)y = xe^x \sin x$.

CF = $(D^2-2D+1)y=0$

AE = m^2-2m+1

$m^2-m-m+1 \Rightarrow m(m-1)-1(m-1)$

roots $-1, 1$

$CF = y = (c_1+c_2x)e^x$

$PI = \frac{1}{D^2-2D+1} xe^x \sin x$

$= \frac{1}{(D-1)^2} e^x x \sin x$

$= e^x \left[\frac{1}{(D+1-1)^2} x \sin x\right]$

$= e^x \left[\frac{1}{D^2} x \sin x\right]$ → mthd 2 double integration

$= e^x \left[x\left(\frac{1}{D^2}\sin x\right) - \frac{2D}{D^2}\cdot\left[\frac{1}{D^2}\sin x\right]\right]$

$= e^x \left[-x \sin x + \frac{2}{D}\left[\sin x\right]\right]$

$= e^x \left[-x \sin x - 2\cos x\right]$

$y = CF + PI$

$y = (c_1+c_2x)e^x + e^x(-x\sin x - 2\cos x)$

HOMOGENEOUS DIFFERENTIAL EQT:-

(Cauchy Euler's Form)

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = X(x)$$

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Put $z = \log x$ then $x = e^z$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz} \Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dz}} \rightarrow (i)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dx}\right) = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dx} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

or $\boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}} \rightarrow (ii)$

$$D_1 = D' = \frac{d}{dz}$$

$x\,Dy = D_1 y$ using (i)

$x^2 D^2 y = D_1(D_1 - 1)y$ using (ii)

$x^3 D^3 y = D_1(D_1 - 1)(D_1 - 2)y$

$x^4 D^4 y = D_1(D_1 - 1)(D_1 - 2)(D_1 - 3)y$

Q1 $(x^2D^2y - xDy + 2y) = x\log x$; $D = \frac{d}{dx}$

Put $z = \log x$; then $x = e^z$

$$D_1 = D' = \frac{d}{dz}$$

Eqn reduced to :- $[D'(D'-1) + D' + 2]\,y = e^z z$

$[D'^2 + 2D' + 2]\,y = e^z z$

CF $= [D'^2 + 2D' + 2]\,y = 0$

AE $= m^2 + 2m + 2$

$$\frac{2 \pm \sqrt{4 - 4(2)}}{2} = \frac{2 \pm \sqrt{4i}}{2} = 1 \pm i$$

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$m = 1 \pm i$

CF :- $y = (C_1 \cos z + C_2 \sin z) e^z$

$y = x [C_1 \cos(\log x) + C_2 \sin(\log x)]$

P.I :- $\frac{1}{D'^2 - 2D' + 2} z e^z = e^z \cdot \frac{1}{(D'+1)^2 - 2(D'+1) + 2} \cdot z$

$= e^z \frac{1}{D'^2 + 1} \cdot z$

PI :- $e^z \frac{1}{1+D'^2} z = e^z [1 + D'^2]^{-1} z$

$= e^z [1 - D'^2 + D'^4 - D'^6 + \dots] z$

$= e^z [z]$

PI :- $x \log x$

Soln is :- $y = x [C_1 \cos \log x + C_2 \sin \log x] + x \log x$

Q2. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 5y = x^2 \sin(\log x)$

Put $z = \log x \Rightarrow x = e^z$

$D' = \frac{d}{dz}$

Eqn reduces to :- $[D'(D'-1) - 3D' + 5] y = e^{2z} \sin z$

CF :- $[D'^2 - 4D' + 5] y = 0$

AE :- $m^2 - 4m + 5 = 0$

$m = \frac{4 \pm \sqrt{16 - 4(5)}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$

CF :- $y = (C_1 \cos z + C_2 \sin z) e^{2z}$

$= x^2 (C_1 \cos(\log x) + C_2 \sin(\log x))$

PI :- $\frac{1}{D'^2 - 4D' + 5} e^{2z} \sin z$

$= e^{2z} \left[\frac{1}{(D'+2)^2 - 4(D'+2) + 5} \sin z \right]$

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$$e^{2z}\left[\frac{1}{D'^2-2D'+4}\sin z\right]$$

$$\frac{e^{2z}}{2}\left[\frac{-1}{D'}\sin z\right] = \frac{e^{2z}\cos z}{2}$$

PI :- $\frac{x^2\cos(\log x)}{2}$

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Q1. $[x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1]\,y = (1+\log x)^2$

$D = \frac{d}{dx}$.

A1. Put $z = \log x$ then, $x = e^x$

new eqn is

$$[D'(D'-1)(D'-2)(D'-3) + 6D'(D'-1)(D'-2)$$
$$+ 9D'(D'-1) + 3D' + 1]\,y = (1+z)^2$$

$D' = \frac{d}{dz}$

$$= [(D'^2 - D)(D'^2 - 3D' - 2D' + 6) + 6D'(D'^2 - 2D' - D' + 2)$$
$$+ 9D'^2 - 9D' + 3D' + 1]\,y = (1+z)^2$$

$$= (D'^4 - 5D'^3 + 6D'^2 - D'^3 + 5D'^2 - 6D) + 6(D'^3 - 3D'^2$$
$$+ 2D') + 9D'^2 - 9D' + 3D' + 1]\,y = (1+z)^2$$

$$= [D'^4 + 2D'^2 + 1]\,y = (1+z)^2$$

$$CF = [D'^4 + 2D'^2 + 1]\,y = (1+z)^2 = 1 + 2z + z^2$$

$$[D'^4 + 2D'^2 + 1]\,y = 0$$

$m^2 = p$

AE :- $m^4 + 2m^2 + 1 = 0$

$p^2 + 2p + 1 = 0$

$(p+1)^2 = 0 \quad p = -1 \Rightarrow m^2 = -1$

$m = \pm i \Rightarrow +i, +i, -i, -i$

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CF :- $y_1 = (C_1 + C_2 z)\cos z + (C_3 + C_4 z)\sin z$

CF :- $y = (C_1 + C_2 \log x)\cos(\log x) + (C_3 + C_4 \log x)\sin(\log x)$

P.I :- $\frac{1}{D'^4 + 2D'^2 + 1}(1 + 2z + z^2)$

$= \left[1 + D'^4 + 2D'^2\right]^{-1}(1 + 2z + z^2)$

$= \left[1 - (D'^4 + 2D'^2) + (D'^4 + 2D'^2)^2 - \cdots\right](1 + 2z + z^2)$

$= 1 + 2z + z^2 - 4 + 0$

$= z^2 + 2z - 3 = 2\log x + (\log x)^2 - 3$

Final soln :-

$y = (C_1 + C_2 \log x)\cos(\log x) + (C_3 + C_4 \log x)\sin(\log x) + 2\log x + (\log x)^2 - 3.$

Q2. $x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

Put $x = e^z$ or $z = \log x$

$D' = \frac{d}{dz}$

$[D'(D'-1)(D'-2) + 2(D)(D'-1) + 2]y = 10(e^z + e^{-z})$

$[D'(D'^2 - 3D' + 2) + 2(D'^2 - D) + 2]y = 10(e^z + e^{-z})$

$[D'^3 - 3D'^2 + 2D' + 2D'^2 - 2D' + 2]y = 10(e^z + e^{-z})$

$[D'^3 - D'^2 + 2]y = 10(e^z + e^{-z})$

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CF :- $[D'^3 - D'^2 + 2]y = 0$

$m^3 - m^2 + 2 = 0$

$m = -1$ is a root

$$
\begin{array}{r|l}
 & m^2 - 2m + 2 \\
m+1 & m^3 - m^2 + 2 \\
 & -m^3 \pm m^2 \\
\hline
 & -2m^2 + 2 \\
 & -2m^2 \quad -2m \\
\hline
 & 2m + 2 \\
 & 2m + 2 \\
\hline
 & x
\end{array}
$$

$m^2 - 2m + 2$

$$\frac{2 \pm \sqrt{4 - 4(2)}}{2} = \frac{2 \pm 2i}{2} = 1 \pm 1i$$

roots :- $-1, 1 + 1i, 1 - 1i$

CF :- $y = C_1 e^{-z} + (C_3 \sin z + C_2 \cos z) e^z$

$= C_1 e^{-\log x} + x(C_2 \cos(\log x) + C_3 \sin(\log x))$

$$PI = \frac{1}{D'^3 - D'^2 + 2} 10(e^z + e^{-z})$$

$$= 10 \cdot \frac{1}{(1)^3 - 1 + 2} e^z + 10 \cdot \frac{z}{3D'^2 - 2D'} e^{-z}$$

$$= \frac{10}{2} e^z + \frac{10}{5} z e^{-z}$$

$= 5e^z + 2ze^{-z} = 5x + 2\log x\,(x^{-1})$

$y = C_1 x^{-1} + x(C_2 \cos(\log x) + C_3 \sin(\log x))$
$+ 5x + \frac{2}{x}\log x$

:- General formula :-

$$\frac{1}{D-\alpha} X(x) = x^{\alpha}\int x^{-\alpha-1} X(x)\,dx$$

$$\frac{1}{D+\alpha} X(x) = x^{-\alpha}\int x^{\alpha-1} X(x)\,dx$$

Q. $[x^2D^2 + 3xD + 1]y = \frac{1}{(1-x)^2}$

$[D'(D'-1) + 3D' + 1]y = 0$:- CF :-

$D'^2 + 2D' + 1 = 0$

$\frac{-2 \pm \sqrt{4-4}}{2} = -1$

$y = (C_1 + C_2 z)e^{-z}$

$CF = y = (C_1 + C_2 \log x)\frac{1}{x}$

$PI = \frac{1}{D'^2 + 2D' + 1} \cdot \frac{1}{(1-x)^2}$

$= \frac{1}{(D'+1)^2} \frac{1}{(1-x)^2}$

$= \frac{1}{(D'+1)}\left[\frac{1}{(D'+1)} \cdot \frac{1}{(1-x)^2}\right]$

$\alpha = 1$

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$$= \frac{1}{D'+1}\, x^{-1} \int x^{1-1} \frac{1}{(1-x)^2}\, dx$$

$$= \frac{1}{D'+1} \cdot x^{-1} \left[\frac{1}{(1-x)} \right]$$

$$= x^{-1} \int x^{1-1} x^{-1} (1-x)^{-1}\, dx$$

$$= x^{-1} \int \frac{1}{x} \cdot \frac{1}{(1-x)}$$

$$\frac{1}{(x)(1-x)} = \frac{A}{x} + \frac{B}{(1-x)}$$

$$1 = A(1-x) + B(x)$$

$\boxed{x=1}$ $\boxed{1=B}$ $\boxed{x=0}$ $\boxed{A=1}$

$$x^{-1} \int \frac{1}{x} + \frac{1}{(1-x)} = x^{-1}\left(\ln x - \ln(1-x) \right)$$

***

Q. $\left[(3x+2)^2 D^2 + 3(3x+2) D - 36 \right] y = 3x^2 + 4x + 1$

Put $v = 3x+2 \rightarrow \frac{dv}{dx} = 3$ $\qquad D = \frac{d}{dx}$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx} = \frac{3dy}{dv} \Rightarrow D = 3D' \quad \left(D' = \frac{d}{dv} \right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3dy}{dv} \right) = 3 \frac{d}{dv}\left(\frac{dy}{dv} \right) \cdot \frac{dv}{dx} = 3^2 \frac{d^2y}{dv^2}$$

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$$\frac{d^2y}{dx^2} = 9\frac{d^2y}{dv^2} \quad (D^2 = 3^2 D'^2)$$

$$[v^2D^2 + 3vD - 36]y = 3x^2+4x+1$$

$$[9v^2D'^2 + 3v(3D') - 36]y = 3\left(\frac{v-2}{3}\right)^2 + 4\left(\frac{v-2}{3}\right)$$

Put $z = \log v \Rightarrow v = e^z$

$$[9E'(E'-1) + \cancel{3}9E' - 36]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right)$$

where $E' = \frac{d}{dz}$

CF :- $[9E'^2 - 9E' + 9E' - 36]y = 0$

$$9E'^2 - 36 = 0$$

$$9E'^2 = 36 \Rightarrow E'^2 = 4$$

$$\boxed{E' = \pm 2}$$

$$y = C_1e^{2z} + C_2e^{-2z}$$

$$= C_1e^{2\log v} + C_2e^{-2\log v}$$

$$= C_1e^{\log v^2} + C_2e^{\log(v^{-2})}$$

$$= C_1v^2 + \frac{C_2}{v^2}$$

$$v = 3x+2 \Rightarrow v^2 = 9x^2+4+12x$$

$$CF = y = C_1(9x^2+4+12x) + \frac{C_2}{(9x^2+4+12x)}$$

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$PI :- \frac{1}{9E'^2-36}\left(3\left(\frac{e^x-2}{3}\right)^2+4\left(\frac{e^x-2}{3}\right)+1\right)$

$= \frac{1}{9}\left(\frac{1}{E'^2-4}\right)\left(\cancel{3}\left(\frac{e^{2x}+4-4e^x}{\cancel{9}\,3}\right)+\frac{4}{3}(e^x-2)+1\right)$

$= \frac{1}{9}\,\frac{1}{E'^2-4}\,\frac{e^{2x}+4-\cancel{4e^x}+\cancel{4e^x}-8+3}{3}$

$= \frac{1}{27}\,\frac{1}{E'^2-4}\,e^{2x}-1$

$= \frac{1}{27}\,\frac{1}{(E'+2)}\left[\frac{1}{(E'-2)}e^{2x}-1\right]$

↓

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WRONSKIAN :-

$y_1(x), y_2(x), \dots, y_n(x)$ are functions of x

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_n^{(1)} \\ y_1^{(2)} & y_2^{(2)} & \cdots & y_n^{(2)} \\ \vdots & & & \\ y_n^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$y' = \frac{dy}{dx}$; $y^2 = \frac{d^2y}{dx^2}$; $y^3 = \frac{d^3y}{dx^3}$; $y^{(n-1)} = \frac{d^{n-1}y}{dx^{n-1}}$

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$W = 0$; then $y_1(x), y_2(x), \dots y_n(x)$ are linearly dependent.

$W \neq 0$; then $y_1(x), y_2(x), \dots y_n(x)$ are linearly Independent.

Q. $y'' - 2y' + y = 0$. find that the solns of given eqn are linearly independent.

$CF = [D^2 - 2D + 1]y = 0$

$m^2 - 2m + 1$

$(m-1)(m-1) = 0$

$m = 1, 1$

$y = (C_1 + C_2 x)e^x = C_1 e^x + C_2 x e^x$

Soln are e^x, xe^x

$y_1 = e^x \qquad y_2 = xe^x$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$$

$$e^{2x} + xe^{2x} - xe^{2x} = e^{2x} \neq 0$$

e^x & xe^x are linearly independent.

METHOD OF VARIATION OF PARAMETERS.

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X(x)$$

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cal. CF :- $y = C_1 y_1(x) + C_2 y_2(x)$

PI :- $y = \underline{\underline{u_1}} y_1 + \underline{\underline{u_2}} y_2$, $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$u_1 = -\int \frac{y_2 X(x)}{W(x)} dx$

$u_2 = \int \frac{y_1 X(x)}{W(x)} dx$

sol^n is $y = C.F + P.I.$

Q1. $y'' - 3y' + 2y = 2$

$CF = [D^2 - 3D + 2] y = 0$

$m^2 - 3m + 2 = 0$
$m^2 - 2m - m + 2$
$m(m-2) - 1(m-2)$
$(m-1)(m-2)$
$m = 1, 2$

$y = C_1 e^x + C_2 e^{2x}$ ∴ CF :-

$y_1 = e^x$; $y_2 = e^{2x}$

$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$

$u_1 = -\int \frac{e^{2x}(2)}{e^{3x}} = -2\int e^{-x} = 2e^{-x}$

$u_2 = \int \frac{e^x(2)}{e^{3x}} = 2\int e^{-2x} = -e^{-2x}$

$PI = 2e^{-x}e^{x} - e^{-2x}.e^{2x} = 2-1 = 1$

$y = c_1 e^{x} + c_2 e^{2x} + 1.$

Q2. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} = e^{x}\sin x$

$[D^2 - 2D]y = 0$

$m^2 - 2m = 0$

$m = 2, 0$

$y = c_1 e^{2x} + c_2 e^{0}$

$y_1 = e^{2x} \quad y_2 = e^{0} = 1$

$W = \begin{vmatrix} e^{2x} & 1 \\ 2e^{2x} & 0 \end{vmatrix} \Rightarrow -2e^{2x}$

$u_1 = -\int \frac{y_2 X(x)}{W(x)} = -\int \frac{e^{x}\sin x}{-2e^{2x}} = \frac{1}{2}\int \frac{\sin x}{e^{x}}$

$\int \frac{\sin x}{e^{x}} = \int \sin x\, e^{-x} = -\sin e^{-x} + \int \cos x\, e^{-x}$

$\int \cos x\, e^{-x} = -\cos x\, e^{-x} - \int \sin x\, e^{-x}$

$\int \frac{\sin x}{e^{x}} = -\sin e^{-x} - \cos x\, e^{-x} - \int \frac{\sin x}{e^{x}}$

$\int \frac{\sin x}{e^{x}} = -\left(\frac{\sin x}{2e^{x}} + \frac{\cos x}{2e^{x}}\right)$

$= \frac{-1}{4e^{x}}(\sin x + \cos x)$

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$$u_2 = \int \frac{y_1 X(x)}{W(x)} = \int \frac{e^{2x} \cdot e^x \sin x}{-2e^{2x}}$$

$$= \frac{-1}{2}\int e^x \sin x$$

$$\int \sin x \, e^x = \sin x \, e^x - \int \cos x \, e^x$$

$$\int \cos x \, e^x = \cos x \, e^x + \int \sin x \, e^x$$

$$\int \sin x \, e^x = \sin x \, e^x - \cos x \, e^x - \int \sin x \, e^x$$

$$\int \sin x \, e^x = \frac{e^x}{2}(\sin x - \cos x)$$

$$u_2 = \frac{-e^x}{4}(\sin x - \cos x)$$

$$P.I = y_1 u_1 + y_2 u_2$$

$$= e^{2x}\left(\frac{-1}{4e^x}(\sin x + \cos x)\right) + \frac{-e^x}{4}(\sin x - \cos x)$$

$$= \frac{-1}{4}\left(e^x(\sin x + \cos x) - e^x(\sin x - \cos x)\right)$$

$$= \frac{-1}{2} e^x \cancel{\cos x} \sin x$$ Ans

$$y = C_1 e^{2x} + C_2 e^{x} \mp \frac{1}{2} e^x \cancel{\cos x} \sin x$$

Q3 $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x$ (Cauchy-Euler's for

Put $z = \log x$; then $x = e^z$

$$D' = \frac{d}{dz}$$

Eq^t reduces to :- $[D'(D'-1) + D' - 1]y = 0$

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$$[D'^2 - D' + D' - 1]\, y = 0$$

$$m^2 = 1$$
$$m = \pm 1$$

$$y = c_1 e^z + c_2 e^{-z}$$
$$= c_1 x + c_2 x^{-1}$$

$y_1 = x$; $y_2 = \frac{1}{x}$ $\quad W = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix}$

$$-\frac{1}{x^2} \cdot x - \frac{1}{x} = -\frac{2}{x}$$

PI :- $u_1 y_1 + u_2 y_2$

$$u_1 = -\int \frac{1/x \, x^2 e^x}{-2/x} dx$$

$$= \frac{1}{2} \int x^2 e^x dx$$

$$\int x^2 e^x = x^2 e^x - \int 2x e^x$$

$$\int 2x e^x = 2 \int x e^x = 2[x e^x - e^x]$$

$$\int x^2 e^x = x^2 e^x - 2x e^x + 2e^x$$

$$u_1 = \frac{1}{2} x^2 e^x - x e^x + e^x$$

$$u_2 = \int \frac{x \cdot x^2 e^x}{-2/x} = \frac{-1}{2} \int x^4 e^x$$

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$$= -\frac{1}{2}\left[x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x\right]$$

$$u_2 = \frac{-x^4e^x}{2} + 2x^3e^x - 6x^2e^x + 12xe^x - 12e^x$$

$$PI = y_1u_1 + y_2u_2$$

$$= \frac{x^3e^x}{\cancel{2}} - x^2e^x + xe^x + -\frac{x^3e^x}{\cancel{2}} + 2x^2e^x - 6xe^x + 12e^x - \frac{12e^x}{x}$$

$$= x^2e^x - 5xe^x + 12e^x - \frac{12e^x}{x}$$

$y = CF + PI$.

$$y = c_1x + c_2x^4 + x^2e^x - 5xe^x + 12e^x - \frac{12e^x}{x}$$ Ans

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MATRIX

$A = [a_{ij}]_{m \times n}$ m = no. of rows
n = no. of column

$$
\begin{array}{cc}
 & \begin{array}{cccc} \text{1st column} & \text{2nd column} & & n^{th}\text{ column} \end{array} \\
A = & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}
\end{array}
\begin{array}{l} \rightarrow \text{1st ROW} \\ \rightarrow \text{2nd ROW} \\ \\ \rightarrow m^{th}\text{ Row} \end{array}
$$

$a_{ij} \in \mathbb{R}, \forall i\,j$ then
A is Real Matrix

If $p + iq = a_{ij} \in \mathbb{C}$ $\cancel{\forall i,j \text{ then}}$

then A is complex Matrix

(a) Row vector = $R = [r_1\ r_2\ \cdots\ r_n]_{1 \times n}$

1 row, n column
Row matrix of order n

(b) Column vector = $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}_{m \times 1}$

m - rows
1 column
Column Matrix of order m

(c) Equal Matrix

$A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{p\times q}$

$A = B \iff m = p$ & $n = q$
and $a_{ij} = b_{ij}, \forall i, j$

(d) Square Matrix (Real or complex both)

if $A_{m\times n}$ st $m = n$
no. of rows = no. of columns

(e) Diagonal Matrix

$$A_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ | & | & & | \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

principal diagonal
entries :- diagonal element
all other :- off Diagonal entries

Diagonal entries are = $a_{ii}, \forall i$
off Diagonal entries are = a_{ij} $\forall i \neq j$

A diagonal matrix A is a square matrix with all off diagonal elements = 0

$$D = A = \begin{bmatrix} \overset{d_1}{a_{11}} & 0 & \cdots & 0 \\ 0 & a_{12}^{\,d_2} & & 0 \\ | & | & \ddots & | \\ 0 & 0 & & \underset{d_n}{a_{nn}} \end{bmatrix}_{n\times n}$$

$D = [d_1 \quad d_2 \quad \cdots \quad d_n]_{n\times n}$

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(f) Scalar Matrix

A diagonal matrix A is said to be a scalar matrix if all diagonal entries are equal to some scalar α.

$a_{11} = a_{22} = \cdots = a_{n} = \alpha$; α is scalar

$$S = \begin{bmatrix} \alpha & & 0 \\ & \alpha \ddots & \\ 0 & & \alpha \end{bmatrix}_{n \times n}$$

(g) Identity Matrix

Identity Matrix is a Scalar matrix where scalar = 1 $\boxed{\alpha = 1}$

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 1 \end{bmatrix}_{n \times n}$$

(h) Null / Zero Matrix

$N = 0$ all the entries are zero. It can be rectangular, Square, Real or Complex matrix.

(i) Transpose of Matrix

$\underset{m \times n}{A}$:- Transpose of A is A^T / A' / A^t
rows $\longrightarrow$ columns
columns $\longrightarrow$ rows

$A_{m \times n} \longrightarrow A^T_{n \times m}$ where $a_{ij} \to a_{ji} \; \forall \; i, j$

Properties of Transpose :- (prove at home)

(a) $(A^T)^T = A$
(b) $(A+B)^T = A^T + B^T$
(c) $(AB)^T = B^T A^T$

(j) Symmetric Matrix
$(A_{m \times m} \in \mathbb{R})$
Any Real Square Matrix A is said to be Symmetric if $A = A^T$
$a_{ij} = a_{ji} \; \forall i,j$

(k) Skew Symmetric Matrix
$A_{m \times m} \in \mathbb{R}$ is said to be skew symmetric if $A = -A^T$
i.e $a_{ij} = -a_{ji} \; \forall i,j$

(l) Orthogonal Matrix
$A_{n \times n} \in \mathbb{R}$ is said to be orthogonal if
$AA^T = A^T A = I$
ie $\boxed{A^T = A^{-1}}$

Properties of (j) & (k) :-

(a) Diagonal entries of a skew sym matrix are always zero.
$a_{ii} = -a_{ii} \; \forall i$
$2a_{ii} = 0 \; \forall i$
$a_{ii} = 0 \; \forall i$

(b) Any Real Sq. matrix A can be expressed as a sum of symmetric & skew symmetric matrix

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$$A_{n\times n} = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$$

Proof (Exam par)

where $A+A^T$ = Symmetric

$$(A+A^T)^T = A^T + (A^T)^T = A^T + A$$

$A-A^T$ = Skew Symmetric

$$(A-A^T)^T = A^T-(A^T)^T = A^T - A = -(A-A^T)$$

(l) Any Real Sq matrix A which is both symmetric & skew symmetric must be a null matrix.

(m) Conjugate Matrix

If A is any complex matrix of order $m\times n$

$$A_{m\times n} \in \mathbb{C}$$

then conjugate Matrix $\bar{A} = [\bar{a_{ij}}]_{m\times n}$ & $\forall i,j$

$a_{ij} = p+iq \qquad \bar{a_{ij}} = p-iq$

(n) Hermitian Matrix

$A_{n\times n} \in \mathbb{C}$

$$A^\theta = A^* = A^H = A = (\bar{A})^T$$

i.e $A = (\bar{A})^T = A^\theta$

i.e

$$a_{ij} = \bar{a_{ji}} \;;\forall i,j$$

(o) Skew Hermitian Matrix

$$A_{n\times n} \in \mathbb{C}$$

$$A = -\bar{A}^T = -A^\theta = -A^* = -A^H$$

i.e $a_{ij} = -\bar{a_{ji}} \;;\forall i,j$

Properties of (θ) & (θ)

(a) If A is Real sq matrix then A^θ is reduces to symmetric matrix & skew hermition reduces to skew symmetric matrix.

(b) In a hermition matrix, diagonal elements are real numbers.

(c) In a skew hermition matrix, the diagonal elements are zero or purely imaginary

(d) For any complex square matrix A, it can be expressed as a sum of hermition & skew hermition matrix

$$\boxed{A = \underbrace{(A+A^\theta)\frac{1}{2}}_{\text{hermition}} + \underbrace{\frac{1}{2}(A-A^\theta)}_{\text{skew hermition}}}$$

$(A+A^\theta)^\theta = A^\theta + A$
$(A-A^\theta)^\theta = -(A^\theta - A^\theta)$ } proof*

(e) $(A^\theta)^\theta = A$

(f) $(A+B)^\theta = A^\theta + B^\theta$

(g) $(AB)^\theta = B^\theta A^\theta$

*

$(A+A^\theta)^\theta = A^\theta + (A^\theta)^\theta = A^\theta + A$

$(A-A^\theta)^\theta = A^\theta - (A^\theta)^\theta = A^\theta - A = -(A-A^\theta)$

(p) Unitary Matrix :-

$A_{n\times n} \in \mathbb{C}$ is said to be unitary if $AA^\theta = A^\theta A = I$

i.e $\boxed{A^{-1} = A^\theta}$ $\quad A^\theta = (\bar{A})^T$

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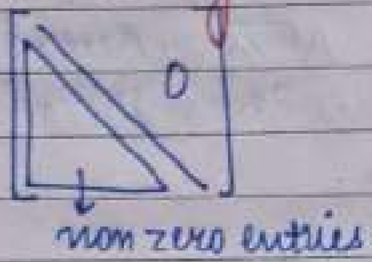
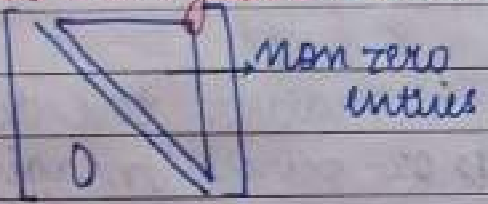
(a) Lower triangular Matrix :-



(b) upper triangular Matrix :-



(3) Minor :-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times4}$$

det of $|A|$ is largest minor
det of sub-matrices are minor
$|X| = |a_{11}|$ = Smallest minor
minor of order 1, 2, 3, 4 exist.

(4) Rank of a Matrix :- (column / row or both)?

$\rho(A) = \text{rank}(A) = r(A) = r$

The rank of a matrix is said to r if there exist
(i) Atleast non-zero minor of order r
(ii) Every minor of order > r must be zero.

a) The rank of a matrix A denotes total no. of linearly independent rows in a matrix A.
or it means

b) Total no. of non-zero rows in a lower triangular matrix or upper triangular matrix. By row & colum interchange we can change it to upper & lower matrix.

Q. Find the rank of

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & -2 \\ 0 & -5 & 7 \end{bmatrix}$$

$R_2 \to R_2 \times \left(-\frac{1}{2}\right)$

$$\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & -5 & 7 \end{bmatrix}$$

$R_3 \to R_3 + 5R_2$

$$\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 12 \end{bmatrix}$$

non-zero rows = 3

$\rho(A) = 3$

Q2. $A = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}_{4\times4}$

upper triangular matrix = Echelon form **

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$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + 3R_1$
$R_4 \to R_4 + 5R_1$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix}_{4\times4} \quad \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 2R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 < 4$

when Rank > order of matrix ⇒ Singular matrix
Singular Matrix = $|A| = 0$

when Rank = Order of matrix ⇒ Non-singular matrix.

Normal Form:-

Any matrix A can be reduced using elementary tranformation can be reduced to following forms called normal forms.

(i) I_r $\qquad \rho(A) = r$

(ii) $\begin{bmatrix} I_r & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$

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(iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow$ 1st canonical form

Both Row & column operations.

favourite (6-12 marks)

Q3. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}_{3\times3}$ Find 2 non-singular matrices P and Q st. PAQ = I and hence find A^{-1}.

$A = I_{3\times3} A I_{3\times3}$

row column Row col

$I = P A Q$

$I^{-1} = (PAQ)^{-1} \Rightarrow I = Q^{-1}A^{-1}P^{-1}$

$QIP = QQ^{-1}A^{-1}P^{-1}P$

$QIP = IA^{-1}I \Rightarrow \boxed{QP = A^{-1}}$

If $A_{5\times4}$

$I_{5\times5} A_{5\times4} I_{4\times4}$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} +1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 + C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$C_3 \to C_3 - 4C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & 3 \end{bmatrix}$$

$$I = P \quad A \quad Q$$

$$\boxed{A^{-1} = QP}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & 3 \end{bmatrix}$$

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INVERSE OF A MATRIX

$A_{n \times n}$, $O(A) = n$ (order)

A is a non-singular matrix

$\Leftrightarrow |A| \neq 0$

$\Leftrightarrow \rho(A) = n$ (order = Rank)

$\Leftrightarrow A^{-1}$ exist

A is a singular matrix

$\Leftrightarrow |A| = 0$

$\Leftrightarrow \rho(A) < n$

$\Leftrightarrow A^{-1}$ doesnot exist

definition

If $A_{n \times n}$ is non-singular square matrix then if there exist a non-singular matrix B of order n st. $AB = BA = I$

then $B = A^{-1}$.

Properties

(a) A^{-1} is unique

(b) $(A^{-1})^{-1} = A$

(c) $(A^{-1})^T = (A^T)^{-1}$

(d) $(AB^{-1}) = B^{-1}A^{-1}$

(e) $(A+B)^{-1} \neq A^{-1} + B^{-1}$.

(f) $(A^{-1})^n = A^{-n} = (A^n)^{-1}$

(g) If $AB = 0$ then $B = 0$ iff A is non singular
or $A = 0$ iff B is non Singular] inverse exist

(h) $AB = AC$ then $B = C \Leftrightarrow A^{-1}$ exist.

(i) If A is a non-Singular symmetric matrix then A^{-1} is also non-singular symmetric matrix.

(j) If A is non-singular upper or lower triangular matrix then A^{-1} is also non-singular upper or lower triangular matrix.

(k) If $D = [d_1, d_2, \ldots d_n]$

then $D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix} = [1/d_1, 1/d_2, \ldots, 1/d_n]$

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$\Rightarrow$ $A^{-1} = \frac{1}{|A|} \text{adj}(A)$

adj (A) = Transpose of cofactors of matrix A.

(**)

$\rightarrow$ Gauss - Jordan Method for Inverse of a Matrix

$$[A_{n\times n} \,|\, I_{n\times n}] \xrightarrow[\text{operations}]{\text{elementary Row}} [I_{n\times n} \,|\, B]$$

Augment

St $AB = BA = I \Rightarrow B = A^{-1}$

Q1. $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \rightarrow R_1 \times -1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 - 3R_1$ $\quad$ $R_3 \rightarrow R_3 + R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 \times 1/2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 + R_2$; $R_3 \to R_3 - 2R_2$.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & \boxed{-5} & -4 & -1 & 1 \end{array}\right]$$

RW

$R_3 \to R_3 \times -1/5$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

$R_1 \to R_1 - \cancel{\frac{2}{3}}\frac{3}{2} R_3$ $\qquad R_2 \to R_2 - \frac{7}{2} R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/10 & 1/5 & 3/10 \\ 0 & 1 & 0 & -13/10 & -1/5 & 7/10 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

St $AB = BA = I$; then $B = A^{-1}$

(prove $AB = BA = I$, then $B = A^{-1}$)

Cramers Rule / System of Equation

$$AX = b$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} , \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

$$a_1 x_1 + a_{12} x_2 + \cdots a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$
$$\vdots$$
$$a_m x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m$$

m :- no. of equation / rows
n :- no. of variables / columns
} → w/o loss of Generalality ($m \geq n$)

(i) <u>Homogeneous</u> :- $b = 0$ i.e $b_i = 0, \forall i$
$AX = 0$ (always consistent)
Trivial solⁿ $X = 0$ always exist

(ii) <u>Non-Homogeneous</u> :- $AX = b$
i.e $b \neq 0$; i.e $\exists$ atleast $b_i \neq 0$, for some i.

System of Equation

- CONSISTENT
  - unique
  - Infinite
- IN-CONSISTENT
  - No-solⁿ

In a consistent sys of Eqⁿ, solⁿ always exist.
In an inconsistent sys of Eqⁿ, solⁿ doesnot exist

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Homogeneous system

$AX = 0$

a) is always consistent

b) Trivial sol $X = 0$ exists

(i) Unique exists

A is non-singular $\boxed{|A| \neq 0}$

A^{-1} exist

$AX = 0$

$A^{-1}(AX) = A^{-1} 0$

$(A^{-1}A) X = 0$

$IX = 0$

$X = 0$

(ii) Infinite soln $\Rightarrow$ non trivial

A is singular

Q. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -2 \\ 4 & 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

"|A| 3×3

$|A|$ $1(12+14) - 2(8+8) + 3(14-12)$

$26 - 32 + 6 = 0$

$|A| = 0 \Rightarrow$ A is singular

$\Rightarrow$ Infinite many soln.

No. of variables to be given arbitary value

= Total no. of variables − Rank of matrix $(\rho(A))$

$= 3 - 2 = 1$.

$x - 16z + 3z = 0$

$x = 13z$

$2x + 4y + 6z = 0$

$2x + 3y - 2z = 0$

$y + 8z = 0$

$y = -8z$

Put $z = t$

$$\left.\begin{array}{l} x + 2y + 3z = 0 \\ 2x + 3y - 2z = 0 \end{array}\right\} \rightarrow \begin{array}{l} x = 13z = 13t \\ y = -8z = -8t \end{array}$$

Non-Homogeneous System

$AX = b$

m :- no. of eqts

n :- no. of variables

Consistent :-

(i) Non-Homogenous system is consistent iff

$\rho(A) = \rho(A|b)$

$A_{m \times n}$ $\quad [A|b]_{m \times n+1}$

Inconsistent :-

$\rho(A) \neq \rho(A|b)$

Unique soln :-

$\rho(A) = \rho(A|b) = n$

Infinite soln :-

$\rho(A) = \rho(A|b) < n$

GAUSS - ELIMINATION :-

Q. Check the consistency & solve

$5x + 3y + 7z = 4$

$3x + 26y + 2z = 9$

$7x + 2y + 10z = 5$

$$[A|b] = \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right] \to \begin{bmatrix} & \\ 0 & \end{bmatrix}$$

$26 - \frac{9}{5}$

$\frac{121}{5}$

$R_2 \to R_2 - R_1 \times \frac{3}{5}$; $R_3 \to R_3 - \frac{7}{5} R_1$

$\frac{10-21}{5}$

$$[A|b] \sim \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & -11/5 & 1/5 & -3/5 \end{array}\right]$$

$R_3 \to R_3 + 1/11 \, R_2$

$$\sim \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Non-zero rows = 2

$\rho(A) = 2$

$\rho(A|b) = 2$

$\rho(A|b) = 2 < 3$

$\rho(A) = 2 < 3$

$\rho(A) = \rho(A|b) = 2 < 3$

consistent & infinite solⁿ.

No. of arbitary values.

$3 - 2 = 1$

Put $z = t$

$$\left.\begin{array}{l} 5x + 3y + 7t = 4 \\ \frac{121y}{5} - \frac{11t}{5} = \frac{33}{5} \end{array}\right\} \to \begin{array}{l} x = \\ y = \end{array}$$

Q2. Find k st

$2x - 3y + 6z - 5t = 3$

$y - 4z + t = 1$

$4x - 5y + 8z - 9t = k$

(i) no solⁿ

(ii) infinite solⁿ.

$$[A|b] = \left[\begin{array}{cccc|c} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 4 & -5 & 8 & -9 & k \end{array}\right]$$

$R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{cccc|c} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 1 & -4 & 1 & k-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{cccc|c} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 0 & 0 & 0 & K-7 \end{array}\right]$$

(i) NO solt :- $\Leftrightarrow \rho(A) \neq \rho(A|b)$

$\rho(A) = 2 < 4$

$\rho(A|b) \neq 2$ i.e when $K-7=0$

$\boxed{K \neq 7}$

(ii) ∞ soltn

$\rho(A) = \rho(A|b) < n$ i.e < 4

i.e when $K-7=0$

$\boxed{K=7}$

$2x - 3y + 6z - 5t = 3$

$1y - 4z + 1t = 1$

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No. of arbitary ~~var~~ $4-2=2$

Put $x=\alpha$, $y=\beta$

Q. Find value of λ & μ.

$2x+3y+5z=9$ (i) unique solt

$7x+3y-2z=8$ (ii) No solt

$2x+3y+\lambda z=\mu$ (iii) ∞ solt.

A. $[A|b] = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$

$R_2 \to R_2 - \frac{7}{2}R_1$

$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15/2 & -39/2 & -47/2 \\ 2 & 3 & \lambda & \mu \end{array}\right]$

$3-\frac{21}{2}$

$\frac{6-21}{2}$

$\frac{-15}{2}$

$-2-\frac{35}{2}$

$\frac{-39}{2}$

$8-\frac{63}{2}$

$\frac{16-63}{2}$

$R_3 \to R_3 - R_1$

$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15/2 & -39/2 & -47/2 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right]$

unique sol^n :- $\rho(A) = \rho(A|b) = n$

$n = 3 = \rho(A) = \rho(A|b)$

~~$\lambda \neq 5$ & $\mu \neq 9$~~ $\lambda - 5 \neq 0$ & no cond^n on μ

$\lambda \neq 5$

No sol^n :- $\rho(A) \neq \rho(A|b)$

~~either $\lambda \neq 5$ or $\mu \neq 9$~~ $\lambda - 5 = 0$ & $\mu - 9 \neq 0$

infinite sol^n

$\lambda - 5 = 0$

& $\mu - 9 = 0$

Put $z = t$

Date: 1-2-2023

Non-Homogeneous System of Equations $AX = b$

Method 1:- Gauss elimination method.

Method 2:- Cramer's Rule.

(for square matrix)

$A_{n\times n}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$

$x_i = \frac{|A_i|}{|A|}$; $\forall\, i = 1, 2, \dots, n$

$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1}$

$$\begin{array}{c} \\ A_i = \end{array} \begin{array}{cccc} & & i\text{th col.} & \\ & & \downarrow & \\ \end{array}$$

$$A_i = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & & b_2 & & a_{2n} \\ \vdots & \vdots & & \vdots & & \\ a_{n1} & a_{n2} & & b_n & \cdots & a_{nn} \end{vmatrix}$$

Conditions

(i) If $|A| \neq 0 \rightarrow$ consistent (unique solⁿ exist)

(ii) If $|A| = 0$ but atleast 1. $|A_i| \neq 0$ for some $i \Rightarrow$ Inconsistent (no solⁿ exist)

(iii) If $|A| = 0$ and $|A_i| = 0\ \forall\, i = 1, 2, \dots, n \Rightarrow$ infinitely many solⁿ exists.

Q.
$4x + 9y + 3z = 6$
$2x + 3y - 3z = 0$
$2x + 6y + z = 2$

$$|A| = \begin{vmatrix} 4 & 9 & 3 \\ 2 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix}$$ ⊗

⊗ $4(1+3) - 9(2+3) + 3(2-1)$

$16 - 45 + 3$ ⊗

$|A| = 19 - 45 = -26$

Q.
$4x + 9y + 3z = 6$
$2x + 3y + z = 2$
$2x + 6y + 2z = 7$

$$|A| = \begin{vmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{vmatrix} = 0$$

$$|A_1| = \begin{vmatrix} 6 & 9 & 3 \\ 2 & 3 & 1 \\ 7 & 6 & 2 \end{vmatrix}$$

$$6(6-6) - 9(4-7) + 3(12-21)$$
$$-9(-3) + 3(-9) = 0$$

$$|A_2| = \begin{vmatrix} 4 & 6 & 3 \\ 2 & 2 & 1 \\ 2 & 7 & 2 \end{vmatrix}$$

$$4(4-7) - 6(4-2) + 3(14-4)$$
$$4(-3) - 6(+2) + 3(10)$$
$$-12 - 12 + 30 = \cancel{30}\,6 \neq 0$$

$$|A_3| = \begin{vmatrix} \cancel{6}\,4 & 9 & 6 \\ \cancel{8}\,2 & 3 & 2 \\ 2 & 6 & 7 \end{vmatrix}$$

$$= 4(21-12) - 9(14-4) + 6(12-6)$$
$$= -18 \neq 0$$

Inconsistent $\Rightarrow$ no soln.

Q2. $x - y + 3z = 3$
$2x + 3y + z = 2$
$3x + 2y + 4z = 5$

$$|A| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix}$$

$$1(12-2) + 1(8-3) + 3(4-9)$$
$$10 + 5 + (-15) = 0$$

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$$|A_1| = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & -1 \\ 5 & 2 & 4 \end{vmatrix} \begin{aligned} &= 3(12-2)+1(8-5)+3(4-15) \\ &= 3(10)+(3)+(3)(-11) \\ &= 30+3-33=0 \end{aligned}$$

$$|A_2| = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{vmatrix} = 0$$

$|A_3| = 0$; Infinite soln

Rank = 2
Arbitary constt = 1

Q. $x-y+3t=3 \quad \times 2 \Rightarrow 2x-2y+6t=6$

$2x+3y+t=2$

$$\cancel{2x}-2y+6t-\cancel{2x}-3y-t=4$$

$$-5y+5t=4$$

$$-5y=4-5t$$

$$y=\frac{5t-4}{5}$$

IIly.

$$x=\frac{11-2t}{5}$$

Q3. $x-y+3z=3$

$2x+3y+z=2$ (same as Q2)

$3x+2y+4z=5$

$$|A| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix} \begin{aligned} &1(12-2)+1(8-3)+3(4-9) \\ &=0 \end{aligned}$$

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Q4. $x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = |A|$$

$1(1+3) + 1(2+3) + 1(2-1)$
$1(4) + 1(5) + 1(1) = 10$

$$|A_1| = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 4(1+3) + 1(+6) + 1(-2)$$
$16 + 6 - 2 = 20$

$$|A_2| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 1(+6) - 4(2+3) + 1(4)$$
$= +6 - 20 + 4 = -10$

$|A_3| = 10$

$$\begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = |A_3|$$

$1(+2) + 1(4) + 4(1)$
$= 10$

$$x_1 = \frac{|A_1|}{|A|} = \frac{20}{10} = 2$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-10}{10} = -1$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{10}{10} = 1$$

Q5. **Linear Transformation :-**

$$Y = AX$$

$Y_{n \times 1}$; $X_{n \times 1}$; $A_{n \times n}$

whenever the vector X transforms into a new vector Y over the matrix A then $Y = AX$ is called Linear Transformation.

Linear property :- for any $\alpha, \beta \in R$, $Y_1 = AX_1$, $Y_2 = AX_2$ then $\alpha Y_1 + \beta Y_2 = A(\alpha X_1 + \beta X_2)$

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(a) A linear transformation is said to regular or non singular transformation if image of distinct vectors x_is are distinct vectors Y_i's

$$X_i = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \xrightarrow{A} Y_i = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

then the matrix A has a property that $\boxed{|A| \neq 0}$

(b) If the images are not distinct then tranformation $Y = AX$ is called a singular tranformation such that the transformation matrix A is also singular $\boxed{|A| = 0}$

(c) A linear transformation $Y = AX$ is called orthogonal transformation if $x_1^2 + x_2^2 + x_3^2 \cdots + x_n^2$ is transformed into $y_1^2 + y_2^2 + \cdots + y_n^2$.

$$X^T I X = X^T X = x_1^2 + x_2^2 + \cdots + x_n^2$$
$$= y_1^2 + y_2^2 + \cdots + y_n^2 \qquad (Y = AX)$$
$$= Y^T Y$$
$$= (AX)^T AX$$
$$= X^T (A^T A) X$$

$$\Rightarrow \boxed{I = A^T A} \rightarrow \text{orthogonal transformation}$$

A is orthogonal matrix.

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Q6. If $y_1 = 5x_1 + 3x_2 + 3x_3$
$y_2 = 3x_1 + 2x_2 - 2x_3$
$y_3 = 2x_1 - x_2 + 2x_3$
Linear transformation from X to Y &

$Z_1 = 4x_1 + 2x_3$
$Z_2 = x_2 + 4x_3$
$Z_3 = 5x_3$

be a Linear transformation from X to Z

Find L.T from Z to Y

As. Hint :- $Y = AX \; (X \to Y)$
$Z = BX \; (X \to Z)$
find Z into Y $\Rightarrow Y = CZ$

$X = B^{-1}Z$
$Y = AX = A(B^{-1}Z) = (AB^{-1})Z$
$C = AB^{-1}$

$$A = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$B^{-1} = \dfrac{\text{adj } B}{|B|}$

$M_{11} = 5 \quad M_{21} = 0 \quad M_{31} = -2$
$M_{12} = 0 \quad M_{22} = 20 \quad M_{32} = 16$
$M_{13} = 0 \quad M_{23} = 0 \quad M_{33} = 4$

$$\text{adj } B = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 20 & -16 \\ 0 & 0 & 4 \end{bmatrix} \rightarrow B^{-1} = \frac{1}{20}\begin{bmatrix} 5 & 0 & -2 \\ 0 & 20 & -16 \\ 0 & 0 & 4 \end{bmatrix}$$

$|B| = 4(5) + 2(0) - 20$

$$AB^{-1} = \frac{1}{20}\begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 0 & -2 \\ 0 & 20 & -16 \\ 0 & 0 & 4 \end{bmatrix}$$

$\exists \rightarrow$ There exist $\quad \in \rightarrow$ belongs
$\Rightarrow \rightarrow$ Implies $\quad \notin \rightarrow$ doesnot belong
$\forall \rightarrow$ for all $\quad \Leftrightarrow \rightarrow$ iff
$\cup \rightarrow$ union $\quad \subset \rightarrow$ subset $\quad \cap \rightarrow$ Intersection

Date :- 2-2-2023

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Linear Dependence

The vectors $x_1, x_2, \ldots, x_n$ are said to be lin dependent if $\exists$ ~~vectors~~ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zeros.
S.t : $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$

Linearly Independent

$x_1, x_2, \ldots x_n$ are called linearly independent if $\exists$ scalars $\alpha_1, \alpha_2, \ldots \alpha_n$ st :- $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$
then each $\alpha_i = 0 \quad \forall i = 1, 2, \ldots, n$

Q. Investigate dep / indep $X_1 = [1, 2, 3]^T$; $X_2 = [3, -2, 1]^T$
$X_3 = [1, -6, -5]^T$

Let there are some scalars $\alpha_1, \alpha_2, \alpha_3$ S.t
$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 = 0$

then $\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix} = 0$

$\alpha_1 + 3\alpha_2 + \alpha_3 = 0$
$2\alpha_1 - 2\alpha_2 - 6\alpha_3 = 0$
$3\alpha_1 + \alpha_2 - 5\alpha_3 = 0$

$\begin{vmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{vmatrix}$ $\quad 1(10+6) - 3(-10+18) + 1(2+6)$
$16 - 3(8) + 8$
$-24 + 24 = 0$

$|A| = 0 \Rightarrow$ Infinite $sol^n \Rightarrow$ non trivial sol^n exist.

Atleast 1 $\alpha_i \neq 0 \ \forall i = 1, 2, 3$

therefore X_1, X_2 & X_3 are linearly dependent to each other.

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$\rho(A) = 2$

Arbitary constt = 1

let $\alpha_3 = t$

$\alpha_1 + 3\alpha_2 + t = 0$ $\times 2$

$2\alpha_1 - 2\alpha_2 - 6t = 0$

$3\alpha_1 + \alpha_2 - 5t = 0$ $\times 3$

$9\alpha_1 + 3\alpha_2 - 15t - \alpha_1 - 3\alpha_2 - t = 0$

$8\alpha_1 - 16t = 0$

$\alpha_1 = \frac{16t}{8} = 2t$

$\cancel{2}\alpha_1 + 6\alpha_2 + 2t - \cancel{2}\alpha_1 + 2\alpha_2 + 6t = 0$

$8\alpha_2 + 8t = 0$

$\boxed{\alpha_2 = -t}$

relnship :- $2tX_1 - tX_2 + tX_3 = 0$

Q2 $X_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ -7 \\ -8 \end{bmatrix}$ $X_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$\alpha_1 + \alpha_2 + 2\alpha_3 = 0$

$3\alpha_1 - 7\alpha_2 + \alpha_3 = 0$

$2\alpha_1 - 8\alpha_2 - \alpha_3 = 0$

$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -7 & 1 \\ 2 & -8 & -1 \end{vmatrix} = 1(+7+8) - 1(-3-2) + 2(-24+14)$

$= 15 + 5 - 20 = \cancel{-20}\ 0$

$|A| = 0 \Rightarrow$ Infinite soln

Non trivial soln

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$\rho(A) = 2$

$x_3 = t$

$x_1 + x_2 + 2t = 0 \quad \times 3 \qquad \times 8$

$3x_1 - 7x_2 + t = 0$

$2x_1 - 8x_2 - t = 0$

$\cancel{3x_1} + 3x_2 + 6t - \cancel{3x_1} + 7x_2 - t = 0$

$10x_2 + 5t = 0$

$$\boxed{x_2 = \frac{-t}{2}}$$

$8x_1 + \cancel{8x_2} + 16t + 2x_1 - \cancel{8x_2} - t = 0$

$10x_1 + 15t = 0$

$$\boxed{x_1 = -\frac{3}{2}t}$$

$-\frac{3}{2}t\,x_1 - \frac{t}{2}x_2 + t\,x_3 = 0$

CHARACTERISTIC EQUATION :-

$A_{n \times n}$, $A \in \mathbb{R}$; $I_{n \times n}$

$\det(A - \lambda I) = 0$ called characteristic Eq^n.

$|A - \lambda I| = 0$

$(-1)^n \lambda^n + (-1)^{n-1} K_1 \lambda^{n-1} + \dots + K_n = 0$ of degree n.

Roots are $\lambda_1, \lambda_2 \cdots \lambda_n$ (n roots)

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Roots are known as characteristic Roots / Latent Roots or Eigen values.

→ For each characteristic root λ, $\exists$ $X \neq 0$ which satisfys $(A - \lambda I) X = 0$ (Homogeneous eqn) ↳ always consistent

$(A - \lambda I) X = 0 \iff |A - \lambda I| = 0$ | we want non trivial sol(n)

ie $\boxed{AX = \lambda X}$

then X is called Eigen vector corresponding to eigen value λ.

Properties of Eigen values

a) If λ is the eigen value for A with corresponding eigen vector x.

$(AX = \lambda X)$ then

(i) λ^m is eigen value for A^m with the same corresponding eigen vector X.

$$\boxed{A^m X = \lambda^m X}$$

(ii) $1/\lambda$ is eigen value of A^{-1} with the same corresponding eigen vector X.

$$AX = \lambda X$$

$$(A^{-1}A)X = \lambda(A^{-1}X)$$

$$\boxed{\frac{1}{\lambda} X = A^{-1} X \quad ; \; X \neq 0}$$

(iii) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values then product of $(\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n) \equiv |A|$

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(iv) The sum of eigen values $\boxed{\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{trace}(A)}$

(v) If ~~αA is~~
$\alpha\lambda$ is eigen value of αA with corresponding eigen vector x

$$\boxed{(\alpha A)x = (\alpha\lambda)x}$$

(vi) $A - KI$ has eigen value $\lambda - K$ with eigen vector x

$$\boxed{(A-KI)x = (\lambda - K)x}$$

(vii) $(A-KI)^{-1}$ has eigen value $\frac{1}{\lambda - K}$ with eigen vector x

ie $$\boxed{(A-KI)^{-1}x = \frac{1}{\lambda-K}x}$$

(viii) A and A^T has same eigen values bcz they have same det.

(ix) If ~~so~~ A is real square matrix & $\alpha + i\beta$ is one of the eigen value the $\alpha - i\beta$ ~~is~~ must be ~~be~~ one ~~of~~ of the eigen values

(x) For a hermitian matrix, eigen values are always real

(xi) For a skew hermitian matrix, eigen values are zeros or purely imaginary

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(xii) for unitary matrix, the eigen values are of magnitude 1.

(xiii) Eigen values for symmetric matrix are always Real.

(xiv) Eigen values for skew-sym are 0 or purely imaginary.

(xv) For orthogonal matrix, Eigen values are of magnitude 1 and either real or complex conjugate pair.

Date :- 7 Feb 2025

Cayley-Hamilton Theorem :-

$$\lambda^n - k_1\lambda^{n-1} + k_2\lambda^{n-2} - \cdots + (-1)^{n-1}k_{n-1}\lambda + k_n = 0$$

$$|A - \lambda I| = 0$$

$$A^n - k_1A^{n-1} + k_2A^{n-2} - \cdots + (-1)^{n-1}k_{n-1}A + k_nI = 0$$

Q. Find characteristic eqn, roots & verify C-H Theorem

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Characteristic Eqt :- $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)((1-\lambda)^2-4) - 2(+\lambda-1-2) = 0$$

$$(1-\lambda)(1+\lambda^2+2\lambda-4) - 2(\lambda-3) = 0$$

$\lambda + \lambda^2 - 2\lambda - 4 - \lambda - \lambda^3 + 2\lambda^2 + 4\lambda - 2\lambda + 5 = 0$

$-\lambda^3 + 3\lambda^2 - \lambda + 3 = 0 \Rightarrow \lambda^3 - 3\lambda^2 + \lambda - 3 = 0$

Let ~~$\lambda=1$~~ $\lambda = 2$ × Root:- 3

~~1+3+1~~ ~~-1+3-1+3~~ $-8+12-2+3$ ×

verify CHT

To show :- $A^3 - 3A^2 + A - 3I = 0$ -(i)

$A^2 =$ $A^3 =$

then put in (i).

for A^{-1} :-

Bcz, By CHT. $A^3 - 3A^2 + A - 3I = 0$

Pre-multiplying A^{-1}:- $A^2 - 3A + I - 3A^{-1} = 0$

$3A^{-1} = A^2 - 3A + I$

$A^{-1} = \frac{1}{3}(A^2 - 3A + I)$

$$A^2 = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & 2+2 & 4 \\ -1-1+2 & -2+1+4 & 2+2 \\ 1-2+1 & 2+2+2 & 4+1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$$

P is not unique

$$A^3 = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1-\cancel{4}+\cancel{4} & -2+4+8 & 8+4 \\ -3+4 & 3+8 & 6+ \\ -6+5 & 6+10 & 12+ \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}$$

Put in (i).

$$A^{-1} = \frac{1}{3}\left(\begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{3}\left(\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix} \right)$$

Gate

Diagonalisation of Matrices

Diagonalisable

Any $A_{n \times n} \in R$ of order n is diagonalisable if $\exists$ an invertible matrix P (MODEL MATRIX) such that $\boxed{P^{-1}AP = D}$

It is also said that the matrix A is similar with the diagonal matrix D $(A \sim D)$ which has eigen values of A as its diagonal elements. $[\lambda_1, \lambda_2 \dots \lambda_n]$

The matrix P is made up of the eigen vectors of corresp. eigen values $(\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n)$

$$P = [X_1, X_2 \ \dots \ X_n]$$

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* The matrix A of order n is diagonalizable $\Leftrightarrow$ it has n linearly independent eigen vectors.

* Under what condition
$A_{n\times n}$ has always n linearly independent eigen vectors when its eigen values are distinct. otherwise also the matrix may have n linearly Independent E. vectors when some of the eigen values are repeated.

$$P^{-1}AP = D$$
$$(PP^{-1})A(PP^{-1}) = PDP^{-1}$$
$$IAI = PDP^{-1}$$
$$\boxed{A = PDP^{-1}}$$

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1}$$
$$= PD^2P^{-1}$$

$$\text{If } D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_n \end{bmatrix} \Rightarrow D^2 = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ 0 & & \ddots \lambda_n^2 \end{bmatrix}$$

$$\boxed{A^n = PD^nP^{-1}} \quad n>0$$

Q. $A = \begin{bmatrix} -2 & 2 & -3 \\ 1 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ $\begin{bmatrix} -2-\lambda & 2 & -3 \\ 1 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0$

$$-2-\lambda((1-\lambda)(-\lambda)-12) - 2(-\lambda-6) + -3(-2-(-1)(1-\lambda))$$
$$-(2+\lambda)(-\lambda+\lambda^2-12) + 2(\lambda+6) - 3(-2+(1-\lambda))$$
$$= -(-2\lambda+2\lambda^2-24-\lambda^2+\lambda^3-12\lambda) + 2\lambda+12$$
$$+6-3+3\lambda$$

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$= \underline{2\lambda} - \underline{2\lambda^2} + \underline{24} + \lambda^2 - \lambda^3 + \underline{12\lambda} + \underline{2\lambda} + \underline{12} + \underline{3} + \underline{3\lambda}$

$-\lambda^3 - \lambda^2 + 19\lambda + 39 = 0$ {#error}

$\lambda^3 + \lambda^2 - \cancel{29}\lambda - \cancel{45} = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\lambda = 2$. $\lambda = -3$

$-8 + 4 + 42 - 45$

$-27 + 9 + 63 - 45 = 0$.

45
27
72

$\lambda = -3$ is one root.

$$\begin{array}{r l}
 & \lambda^2 - 2\lambda - 15 \\
\lambda + 3 \,) & \lambda^3 + \lambda^2 - 21\lambda - 45 \\
 & \lambda^3 + 3\lambda^2 \\
 & -2\lambda^2 - 21\lambda \\
 & -2\lambda^2 - 6\lambda \\
 & -15\lambda - 45 \\
 & -15\lambda - 45 \\
 & \times
\end{array}$$

$\lambda^2 - 2\lambda - 15$

$\lambda^2 - 5\lambda + 3\lambda - 15$

$\lambda(\lambda - 5) + 3(\lambda - 5)$

$(\lambda + 3)(\lambda - 5)$

$\lambda = -3, 5$.

$\underline{\lambda = 5, -3, -3}$ (not diff)

$\boxed{\lambda = -3}$

$[A - \lambda I]\, x = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

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$\rho(A+3I) = 1$

linearly independent eigen vector exists $= n-\rho$
$= 3-1 = 2$

$\Rightarrow$ 2 eigen vector for $\lambda = -3$ (one from $\lambda = 5$)

(1) $x_3 = 1,\ x_2 = -1 \Rightarrow x_1 = ?$
(2) $x_3 = -1,\ x_2 = 2 \Rightarrow x_1 = ?$

$x_1 + 2x_2 - 3x_3 = 0$
$x_1 - 2 - 3 = 0$
$x_1 = 5$

$x_1 + 4 + 3 = 0$
$x_1 = 7$

$X_1 = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$ $\quad X_2 = \begin{bmatrix} 7 \\ 2 \\ -1 \end{bmatrix}$

for $\lambda = 5$
$[A - 5I]X = 0$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\rho(A - 5I) = 2$

eigen vectors $= n - \rho = 3 - 2 = 1$
Say $x_3 = 1 \Rightarrow x_1$ & $x_2 = ?$

$2x_1 - 4x_2 - 6x_3 = 0$
$2x_1 - 4x_2 - 6 = 0$

$-x_1 - 2x_2 - 5 = 0 \times 2$
$-2x_1 - 4x_2 - 10 = 0$

$-8x_2 - 16 = 0$
$-8x_2 = 16$
$\boxed{x_2 = -2}$

$-x_1 + 4 - 5 = 0$
$-x_1 - 1 = 0$
$-x_1 = 1 \Rightarrow \boxed{x_1 = -1}$

$$X_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 5 & 7 & 1 \\ -1 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

order of eigen vector
→ order of eigen values.

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Q11. Find matrix A whose eigen values are 2, 2, 4 & given vectors are $(-2,1,0)^T$, $(-1,0,1)^T$, $(1,0,1)^T$

Ans. $P^{-1}AP = D = [2, 2, 4]$

$$P = [X_1\ X_2\ X_3] = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$A = PDP^{-1}$

Q12. A has eigen values $\overset{\lambda_1}{1}, \overset{\lambda_2}{-1}, \overset{\lambda_3}{2}, \overset{\lambda_4}{-2}$, Find det(B) where $B = 2A + A^{-1} - I$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \rightarrow$ 4×4 matrix (B, I)

$$I_{4\times4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \lambda = 1, 1, 1, 1$$

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Let $\mu_1, \mu_2, \mu_3, \mu_4$ be the eigen values of B

$$\mu_1 = 2\lambda_1 + \frac{1}{\lambda_1} - 1$$

$$\mu_2 = 2\lambda_2 + \frac{1}{\lambda_2} - 1$$

$$\mu_3 = 2\lambda_3 + \frac{1}{\lambda_3} - 1$$

$$\mu_4 = 2\lambda_4 + \frac{1}{\lambda_4} - 1$$

$$|B| = \mu_1 \cdot \mu_2 \cdot \mu_3 \cdot \mu_4$$

$$\text{trace}(B) = \mu_1 + \mu_2 + \mu_3 + \mu_4$$

⇒ Quadratic forms :-

$$Q = X^T A X$$

for $A_{n\times n}$, The quadratic form Q is given by $X^T A X$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

Homogeneous expression in x (Power 2 of each term)

$$= a_{11} x_1^2 + (a_{12} + a_{21}) x_1 x_2 + (a_{13} + a_{31}) x_1 x_3 + \cdots$$
$$+ a_{22} x_2^2 + \cdots + a_{nn} x_n^2 + (a_{1n} + a_{n1}) x_1 x_n + \cdots$$

$$Q = X^T A X$$

in the form of symmetric matrix

$$C = [c_{ij}]$$

where $c_{ij} = \frac{a_{ij} + a_{ji}}{2}$; $\forall i, j$

and $c_{ij} = c_{ji}$

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then $Q = X^T C X$; C: Symmetric

Q. $Q = x_1^2 + 3x_1x_2 + x_2^2$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$c_{11} = a_{11}$

$c_{22} = a_{22}$

$c_{12} = \frac{a_{12} + a_{21}}{2} = \frac{3}{2}$

$c_{12} = c_{21} = \frac{3}{2}$

$$C = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 1 \end{bmatrix}$$

$$Q = x_1^2 + \frac{3}{2}x_1x_2 + \frac{3}{2}x_2x_1 + x_2^2$$

Standard / Canonical form

If A is symmetric matrix with E. values $\lambda_1, \lambda_2, \ldots \lambda_n$ such that $Q = X^T A X$ and P = any orthogonal matrix (normalised form) s.t. A is diagonalisable and ∃ a transformation $X = PY$ then

$Q = X^T A X$

$= (PY)^T A (PY)$

$= Y^T (P^T A P) Y$ $\quad (P^T = P^{-1}$ orthogonal$)$

$= Y^T D Y$ $\quad (P^T A P = D)$

canonical form $= \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$

Q Find canonical form for $Q = 17x_1^2 + 30x_1x_2 + 17x_2^2 = 128$. What type of conic section does it represent. Also find L.T & matrix transformation.

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$A = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$

$\begin{array}{r} 17 \\ \times 17 \\ \hline 119 \\ 17\cdot \\ \hline 289 \end{array}$

$|A - \lambda I| = 0$ $\quad \begin{vmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{vmatrix}$

$(17-\lambda)^2 - (-15)^2 = 0$

$289 - 34\lambda + \lambda^2 - 225 = 0$

$\lambda^2 - 34 + 64 = 0$

$\lambda = 2, 32.$

$\begin{array}{r} 289 \\ -225 \\ \hline 64 \end{array}$

$\lambda = 2$

$[A - \lambda I]X = 0 \Rightarrow \underbrace{\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}}_{B} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

ie $[A - 2I]X = 0$

$\rho(B) = 1, \quad x_1 = 1 \Rightarrow x_2 = 1 \quad \Rightarrow X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Normalised $\longrightarrow \hat{X}_1 = \begin{vmatrix} 1/\sqrt{1^2+1^2} \\ 1/\sqrt{1^2+1^2} \end{vmatrix} = \begin{vmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{vmatrix}$

normalised

$x_i \to \dfrac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots x_n^2}}$

$\lambda = 32$

$[A - 32I]X = 0 \Rightarrow \underbrace{\begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\rho(C) = 1 \quad x_1 = 1 ; x_2 = -1$

$X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

normalised :- $\hat{X}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad x = PY \downarrow \boxed{\text{rest all}}$$

Standard form :- $Q = 2x_1^2 + 32x_2^2 = 128$

$$\frac{x_1^2}{64} + \frac{x_2^2}{4} = 1 \Rightarrow \text{ellipse}$$

Q. $Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2 = 144$

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 5 & -2 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix}$$

$$3-\lambda((5-\lambda)(3-\lambda) - (1)) + 1((-1)(3-\lambda) + 1) + 1(1 - (5-\lambda))$$

$$= 3-\lambda(15 - 5\lambda - 3\lambda + \lambda^2 - 1) + 1(\lambda - 3 + 1) + (1 - 5 + \lambda)$$

$$= 3-\lambda(\lambda^2 - 8\lambda + 14) + (2\lambda - 6)$$

$$= 3\lambda^2 - 24\lambda + 42 - \lambda^3 + 8\lambda^2 - 14\lambda + 2\lambda - 6$$

$$= -\lambda^3 + 11\lambda^2 \mp 36\lambda + 36 = 0$$

$$= \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2$$

$$8 - 44 + 72 - 36 = 0$$

$$\lambda - 2 \div \lambda^3 - 11\lambda^2 + 36\lambda - 36$$

$$
\begin{array}{r}
\lambda^2 - 9\lambda + 18 \\
\lambda - 2 \,\big)\, \overline{\lambda^3 - 11\lambda^2 + 36\lambda - 36} \\
\underline{-\lambda^3 \mp 2\lambda^2} \qquad\qquad\qquad \\
-9\lambda^2 + 36\lambda \qquad \\
\underline{-9\lambda^2 + 18\lambda} \qquad \\
+18\lambda - 36 \\
\underline{18\lambda - 36} \\
\underline{\times}
\end{array}
$$

$\lambda^2 - 9\lambda + 18$

$\lambda^2 - 6\lambda - 3\lambda + 18$

$\lambda(\lambda - 6) - 3(\lambda - 6)$

$(\lambda - 3)(\lambda - 6) \Rightarrow \boxed{\lambda = 3, 6}$

$\lambda = 2$

$|A - 2I| = 0 \Rightarrow \begin{vmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$1(3-1) + 1(-1+1) + 1(1-3) = 0$

$\rho(B) = 2 \quad x_1 = 1, \; x_2 = 2, \; x_3 - 1$

$1 - 2 + x_3 = 0$

$x_3 = 1$

$|A - 3I| = 0 \Rightarrow \begin{vmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{vmatrix}$

$1(1) + 1(1-2) = 0$

$\rho(B) = 2$

$x_1 = 1, \; x_2 = 2$

$-1 + 4 - x_3 = 0$

$\boxed{3 = x_3}$

$|A - 6I| = 0 \Rightarrow \begin{vmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{vmatrix}$

$-3(3-1) + 1(3+1) + 1(1+1)$

$= -3(2) + 4 + 2$

$= 0$

$x_1 = 1$

$x_2 = 3$

$x_3 = 6$

$\rho(B) = 2 \quad x_1 = 1, \; x_2 = 2$

$-3x_3 = 1 \Rightarrow \boxed{x_3 = -1/3} \quad 1 - 2 - 3x_3 = 0$

$$\hat{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$\hat{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

$$1 + 4 + \frac{1}{9}$$

$$46/9$$

$$\hat{x}_3 = \begin{bmatrix} \cancel{1} \\ \cancel{2} \\ \cancel{1/3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{46} \\ 3/\sqrt{46} \\ 6/\sqrt{46} \end{bmatrix}$$

$X = PY$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{14} & 1/\sqrt{46} \\ 2/\sqrt{6} & 2/\sqrt{14} & 3/\sqrt{46} \\ 1/\sqrt{6} & 3/\sqrt{14} & 6/\sqrt{46} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = \frac{y_1}{\sqrt{6}} + \frac{y_2}{\sqrt{14}} + \frac{y_3}{\sqrt{46}}$$

$$x_2 = \frac{2}{\sqrt{6}} y_1 + \frac{2}{\sqrt{14}} y_2 + \frac{3}{\sqrt{46}} y_3$$

$$x_3 = \frac{y_1}{\sqrt{6}} + \frac{3}{\sqrt{14}} y_2 + \frac{6}{\sqrt{46}} y_3$$

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VECTORS

∇-operator

$$= \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$\phi(x,y,z) \rightarrow$ scalar

$V(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$
(vector)

Gradient :- $\nabla\phi$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x,y,z)$$

$$= \hat{i}\frac{\partial}{\partial x}\phi + \hat{j}\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

scalar $\longrightarrow$ vector

Divergence $= \vec{\nabla}\cdot\vec{V}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

$$\text{div}(\vec{v}) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

$\hat{i}\cdot\hat{i} = 1$; $\hat{i}\cdot\hat{j} = 0 = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i}$ $\quad\hat{j}\cdot\hat{j} = 1 = \hat{k}\cdot\hat{k}$

Curl $(\vec{v}) = \vec{\nabla}\times\vec{V}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

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NORMAL :-

$$d\phi = \frac{\partial\phi}{\partial x}\cdot dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \underbrace{\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)}_{\nabla\cdot\phi}\cdot\left(\underbrace{\hat{i}dx + \hat{j}dy + \hat{k}dz}_{d\vec{r}}\right)$$

$\vec{x} = (x_1, x_2, x_3)$
$= x_1\hat{i} + x_2\hat{j} + x_3\hat{k})$

$\vec{r} = (x, y, z)$
$= x\hat{i} + y\hat{j} + z\hat{k}$

$$d\vec{x} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$d\phi = \vec{\nabla}\phi\cdot d\vec{r}$$

$$d\phi = |\vec{\nabla}\phi|\cdot|d\vec{r}|\cos\theta$$

max $\theta = 0$ $\quad \parallel^{x}$

$\theta = \pi/2 \quad \perp \Rightarrow \vec{\nabla}\phi$ - Normal to curve
$\phi(x, y, z) = c$

unit Normal :- $\dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$

DIRECTIONAL DERIVATIVE :-

In the direction of vector D for $\vec{\nabla}\phi$ is given by
$\vec{\nabla}\phi\cdot\hat{a}$; $\hat{a} = \dfrac{\vec{d}}{|\vec{a}|}$

Q The temp at any pt in the space is given by $T = xy + yz + zx$. Determine the directional derivative of T in the dir of vector $\vec{d} = 3\hat{i} - 4\hat{k}$ at the point (1,1,1).

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$$\nabla \cdot T = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)T$$
$$= \hat{i}(y+z) + \hat{j}(x+z) + \hat{k}(x+y)$$

$$\nabla T\Big|_{(1,1,1)} = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$DD = \vec{\nabla}T \cdot \hat{a} = (2\hat{i} + 2\hat{j} + 2\hat{k})\left(\frac{3\hat{i} - 4\hat{k}}{5}\right)$$
$$= \frac{6}{5} - \frac{8}{5} = \frac{-2}{5}$$

Q. If the fluid $(\vec{F})$ is compressible $\vec{F} = (F_1, F_2, F_3)$ then $\operatorname{div}(\vec{F}) = 0 = \vec{\nabla} \cdot \vec{F}$ & F is called SOLENOIDAL then $\exists$ a scalar qty $\phi(x,y,z)$ called scalar Potential s.t $\vec{F} = \vec{\nabla}\phi$

Q1. Find the value of n for which $r^n \vec{r}$ is solenoidal

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad |\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$
$$r^n = (x^2+y^2+z^2)^{n/2}$$

$$r^n\vec{r} = x(x^2+y^2+z^2)^{n/2}\hat{i} + y(x^2+y^2+z^2)^{n/2}\hat{j} + z(x^2+y^2+z^2)^{n/2}\hat{k}$$

Given $r^n\vec{r}$ is solenoidal

$$\operatorname{div}(r^n\vec{r}) = 0$$

$$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \left[x(x^2+y^2+z^2)^{n/2}\hat{i} + y(x^2+y^2+z^2)^{n/2}\hat{j} + z(x^2+y^2+z^2)^{n/2}\hat{k}\right] = 0$$

$$(x^2+y^2+z^2)^{n/2} + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot(2x^2) + (x^2+y^2+z^2)^{n/2}$$
$$+ \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot(2y^2) + (x^2+y^2+z^2)^{\frac{n}{2}} + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2z^2)$$

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$$(x^2+y^2+z^2)^{\frac{n}{2}-1}\left(3x^2\,3y^2\,3z^2+n(x^2+y^2+z^2)\right)=0$$

$$(x^2+y^2+z^2)^{\frac{n}{2}-1}\left(3x^2+3y^2+3z^2+n(x^2+y^2+z^2)\right)=0$$

$$(x^2+y^2+z^2)^{\frac{n}{2}-1}\underbrace{(x^2+y^2+z^2)}_{\text{non zero.}}(3+n)=0$$

$$\boxed{n=-3}$$

If $\vec{\nabla}\times\vec{V}=0$ (curl $\vec{V}=0$) ; $\vec{V}$ is irrotational then $\exists$ a scalar ϕ st $\vec{V}=\vec{\nabla}\phi$.

Q. Show that the vector $\vec{V}=2xyz\hat{i}+(x^2z+2y)\hat{j}+x^2y\hat{k}$ is irrotational.
& find scalar pot u.

$$\vec{V}=2xyz\hat{i}+(x^2z+2y)\hat{j}+x^2y\hat{k}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z+2y & x^2y \end{vmatrix}$$

$$\hat{i}\left(\frac{\partial}{\partial y}(x^2y)-\frac{\partial}{\partial z}(x^2z+2y)\right)-\hat{j}\left(\frac{\partial}{\partial x}(x^2y)-\frac{\partial}{\partial z}(2xyz)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(x^2z+2y)-\frac{\partial}{\partial y}(2xyz)\right)$$

$$=\hat{i}(x^2-x^2)-\hat{j}(2xy-2xy)+\hat{k}(2zx-2zx)$$

$$=0$$

To find $u(x,y,z)$

$$d\vec{u} = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right)\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$$

$$= \vec{\nabla}u \cdot d\vec{r}$$

because $\text{curl}\,\vec{v} = 0 \; \therefore \; \exists \; u(x,y,z)$ called scalar pot s.t $\vec{v} = \vec{\nabla}u$

$$d\vec{u} = \vec{v} \cdot d\vec{r}$$

$$= [2xyz\hat{i} + (x^2z + 2y)\hat{j} + x^2y\hat{k}] \cdot [\hat{i}dx + \hat{j}dy + \hat{k}dz]$$

$$= 2xyz\,dx + (x^2z + 2y)dy + x^2y\,dz$$

$$d\vec{u} = yz(2x\,dx) + x^2z\,dy + 2y\,dy + x^2y\,dz$$

$$\cancel{\int d\vec{u} = yz\int 2x\,dx + x^2z\int dy + \int 2y\,dy + x^2y\int dz}$$

$$= yz\,d(x^2) + x^2z\,dy + d(y^2) + x^2y\,dz$$

$$= z[y\,d(x^2) + x^2dy] + d(y^2) + x^2y\,dz$$

$$= z[d(x^2y)] + x^2y\,dz + d(y^2)$$

$$du = d(x^2yz) + d(y^2)$$

$$du = d[x^2yz + y^2]$$

$$u = x^2yz + y^2 + c$$

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Q. $\vec{F} = (y^2 + 2xz^2)\hat{i} + (2xy - z)\hat{j} + (2x^2z - y + 2z)\hat{k}$
irrotational & scalar potential

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+2xz^2 & 2xy-z & 2x^2z-y+2z \end{vmatrix}$$

$$\hat{i}\left(\frac{\partial}{\partial y}(2x^2z-y+2z) - \frac{\partial}{\partial z}(2xy-z)\right)$$
$$-\hat{j}\left(\frac{\partial}{\partial x}(2x^2z-y+2z) - \frac{\partial}{\partial z}(y^2+2xz^2)\right)$$
$$+\hat{k}\left(\frac{\partial}{\partial x}(2xy-z) - \frac{\partial}{\partial y}(y^2+2xz^2)\right)$$

$$= \hat{i}(-1+1) - \hat{j}(4xz - 4zx) + \hat{k}(2y - 2y)$$

$$= 0$$

$$\vec{du} = \vec{v} \cdot \vec{dr}$$
$$= (y^2+2xz^2)\,dx + (2xy-z)\,dy + (2x^2z - y + 2z)\,dz$$
$$= y^2 dx + z^2(2x\,dx) + (2y)(dy)x - z\,dy + x^2(2z\,dz) - y\,dz + (2z)\,dz$$

$$= y^2 dx + z^2 d(x^2) + d(y^2)x - z\,dy + x^2 d(z^2) - y\,dz + d(z^2)$$

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INTEGRATION

A force acts on a particle and moves it $\widehat{AB}$

$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\text{work done} = \int_A^B \vec{F}\cdot\vec{dr}$$

$$= \oint \vec{F}\cdot\vec{dr}$$

Closed path

$\int \vec{F}\cdot dr = 0$ then $\vec{F}$ is conservative (no work is done)

$\vec{F}$ is irrotational

i.e $\vec{\nabla}\times\vec{F} = 0$

$\exists$ a scalar $\phi(x,y,z)$ st $\vec{F} = \nabla\phi$

Q1 Show that integral $\int_{(1,2)}^{(3,4)} (xy^2 + y^3)dx + (x^2y + 3xy^2)dy$ independent of the path taken, hence find scalar pot.

Ans It means curl $\vec{F} = 0$

$$\vec{F} = (xy^2 + y^3)\hat{i} + (x^2y + 3xy^2)\hat{j}$$

curl F =

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2+y^3 & x^2y+3xy^2 & 0 \end{vmatrix}$$

$$\hat{i}\left(-\frac{\partial}{\partial z}(x^2y+3xy^2)\right) - \hat{j}\left(\frac{\partial}{\partial z}(xy^2+y^3)\right) = 0$$

$\therefore$ Independent of path taken.

$\exists$ scalar $\phi(x,y,z)$ st $\vec{F} = \nabla\phi$

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$$\cancel{I = \int_{(1,2)}^{(3,4)} (xy^2 + y^3)dx + (x^2y + 3xy^2)dy}$$

$$I = \int_{(1,2)}^{(3,4)} (xy^2 dx) + y^3 dx + x^2y\,dy + 3xy^2 dy$$

$$= \int_{(1,2)}^{(3,4)} \qquad \vec{F} = \vec{\nabla}\phi$$

$$d\phi = \vec{\nabla}\phi \cdot \vec{dr}$$

$$d\phi = \vec{F} \cdot \vec{dr}$$

$$= ((xy^2 + y^3)\hat{i} + (x^2y + 3xy^2)\hat{j}) \cdot (\hat{i}dx + \hat{j}dy)$$

$$= (xy^2 + y^3)dx + (x^2y + 3xy^2)dy$$

$$= (xy^2 dx) + y^3 dx + x^2y\,dy + 3xy^2 dy$$

$$= d\left(\frac{x^2}{2}\right)y^2 + y^3 dx + d\left(\frac{y^2}{2}\right)x^2 + d\left(\frac{y^3}{3}\right)3x$$

$$\cancel{= y^2\left(d\left(\frac{x^2}{2}\right) + y\,dx\right) + x^2\left(d\left(\frac{y^2}{2}\right)x + d\left(\frac{y^3}{3}\right)3x\right)}$$

$$= d\left(\frac{x^2}{2}\right)y^2 + d\left(\frac{y^3}{3}\right)3x + y^3 dx + d\left(\frac{y^2}{2}\right)x^2$$

x^3y^3
$3x^2y^3$
$+ x^3 3y^2$

$$= d\left(\frac{x^2}{2}\right)y^2 + d(y^3)x + y^3 dx + d\left(\frac{y^2}{2}\right)x^2$$

$$d\phi = d\left(\frac{x^2y^2}{2}\right) + d(xy)^3$$

$$\phi = \int_{(1,2)}^{(3,4)} d\left(\frac{x^2y^2}{2} + xy^3\right)$$

$$= \frac{(3)^2(4)^2}{2} + (3)(4)^3 - \frac{(1^2)(2)^2}{2} - (1)(2)^3$$

$$= \frac{9\times\cancel{16}^{8}}{\cancel{2}} + 9(64) - \frac{(4)}{2} - (8)$$

$$= 72 + 576 - 10 = 72 + 566 = 254\cancel{8}$$

Surface Integral :-

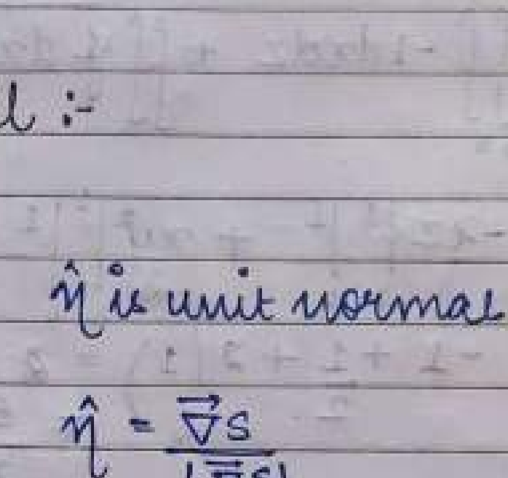
$\vec{F}$ on surface S

$S = \iint \vec{F}\cdot\hat{n}\, ds$ $\quad \hat{n}$ is unit normal to the S.

$$\hat{n} = \frac{\vec{\nabla} S}{|\vec{\nabla} S|}$$

**

Q. Find surface integral where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ & S :- cube bounded by $x=0, x=1$; $y=0$; $y=1$; $z=0, z=1$



	Face	Eqt of surface	dS	$\hat{n}$
1	OABC	$z=0$	$dx\,dy$	$-\hat{k}$
2	BGCD	$y=1$	$dx\,dz$	$\hat{j}$
3	GDEF	$z=1$	$dx\,dy$	$\hat{k}$
4	FAOE	$y=0$	$dx\,dz$	$-\hat{j}$
5	ABGF	$x=1$	$dy\,dz$	$\hat{i}$
6	OCDE	$x=0$	$dy\,dz$	$-\hat{i}$

$$\iint \vec{F}\cdot\hat{n}\,ds = \left[\iint_{OABC} + \iint_{BGDC} + \iint_{GDEF} + \iint_{ABGF} + \iint_{OCDE} + \iint_{AOEF}\right] \vec{F}\cdot\hat{n}\,ds$$

$$= \iint (-y^2\hat{j})\cdot -\hat{k}\,dx\,dy \xrightarrow{} 0 \; + \int_{z=0}\int_{x=0} (4xz\hat{i} - \hat{j} + z\hat{k})\,\hat{j}\,dx\,dz$$

$$+ \int_{x=0}^{1}\int_{y=0}^{1} (4x\hat{i} - y^2\hat{j} + y\hat{k}) \cdot \hat{k}\, dx\,dy + \int_{y=0}^{1}\int_{z=0}^{1} (4z\hat{i} - y^2\hat{j} + yz\hat{k}) \cdot \hat{i}\, dy\,dz$$

$$+ \iint (-y^2\hat{j} + \cancel{yz\hat{k}}^{\,0}) \cdot \hat{i}\, dy\,dz + \iint (4\cancel{xz\hat{i}}^{\,0}) \cdot -\hat{j}\, dx\,dz$$

$$= \int_0^1\int_0^1 -1\, dx\,dz + \iint y\, dx\,dy + \iint 4z^2\, dy\,dz$$

$$= -xz\Big|_0^1\Big|_0^1 + \frac{xy^2}{2}\Big|_0^1\Big|_0^1 + \frac{4z^2}{2} \cdot y\Big|_0^1\Big|_0^1$$

$$= -1 + \frac{1}{2} + 2(1) = 2 \quad \cancel{\text{Ans}}$$

$$= \frac{5}{2} - 1 = \frac{3}{2} \text{ Ans.}$$

GREEN'S THEOREM

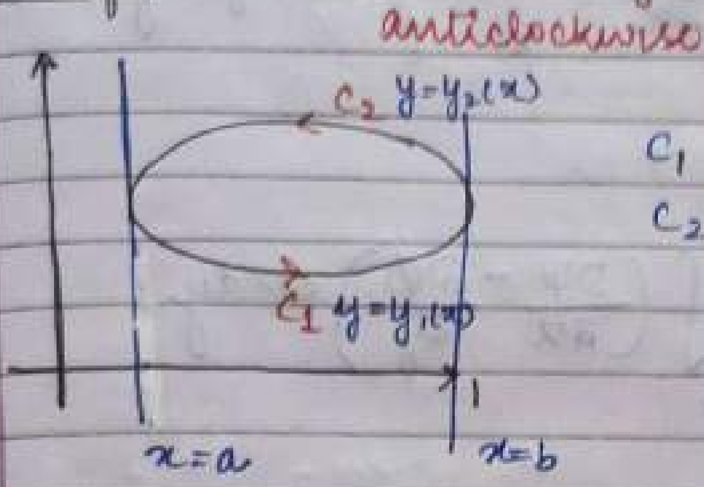
★★ Imp — green's theorem relates line integral with double integral (xy plane) whereas Stokes theorem relates line integral with surface integral in xyz plane.

If $\phi(x,y)$; $\psi(x,y)$; $\frac{\partial \phi}{\partial y}$; $\frac{\partial \psi}{\partial x}$ are continuous functions over a region R bounded by a simple closed curve C in xy plane then line integral over C

$$\int_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F} \cdot \hat{k}\, d\vec{R} \qquad (dR = dx\,dy)$$

Proof :-



$C_1 : \; a \le x \le b \; ; y = y_1(x)$
$C_2 : \; b \le x \le a \; ; y = y_2(x)$

$$\iint_R \frac{\partial \phi}{\partial y} dx dy = \iint_R \frac{\partial \phi}{\partial y} dx dy = \int_{x=a}^{x=b}\int_{y_1(x)}^{y_2(x)} \frac{\partial \phi}{\partial y} dx dy + \int_{x=b}^{x=a}\int_{y=y_1(x)}^{y_2(x)} \frac{\partial \phi}{\partial y} dx dy$$

$$\iint_R \frac{\partial \phi}{\partial y} dx dy = \int_{x=a}^{b} \phi(x,y)\Big|_{y_1(x)}^{y_2(x)} dx = \int_{x=a}^{b} \phi(x,y_2(x)) - \phi(x,y_1(x)) \, dx$$

$$= \int_{x=a}^{x=b} \phi(x,y_2(x)) dx - \int_{x=a}^{b} \phi(x,y_1(x)) dx$$

$$= \underbrace{-\int_{x=b}^{a} \phi(x,y_2(x)) dx}_{C_2} - \int_{x=a}^{b} \underbrace{\phi(x,y_1(x)) dx}_{C_1}$$

$$= -\left[\int_{C_2} \phi(x,y) dx + \int \phi(x,y) dx\right]$$

$$= -\left[\left(\int_{C_2} + \int_{C_1}\right) \phi(x,y) dx\right] = -\int_C \phi(x,y) dx \quad \text{—(i)}$$

Similarly, $\iint_R \frac{\partial \psi}{\partial x}\, dx\, dy = \int_c \psi(x,y)\, dy$ —(ii)

(ii) + (i)

$$\int_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

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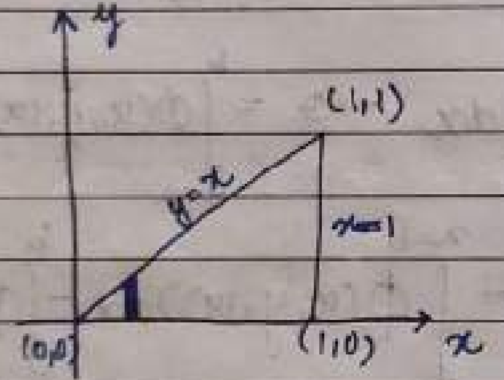
Q. $\int_C x^2 y\, dx + x^2 dy$; C = Δ with vertices (0,0), (1,0) & (1,1)



$\phi(x,y) = x^2 y$

$\psi(x,y) = x^2$

$$\int_C \phi\, dx + \psi\, dy = \int_{x=0}^{1}\int_{y=0}^{x} (2x - x^2)\, dx\, dy$$

$$= \int_{x=0}^{1} (2x - x^2)\, dx \int_{y=0}^{x} dy$$

$x^3, \frac{x^3}{3}$

$$= \int_{x=0}^{1} (2x^2 - x^3)\, dx = \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1$$

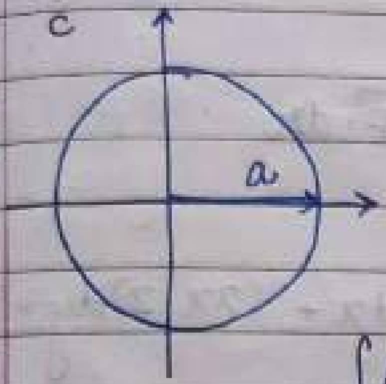
$$= \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}$$ Ans.

$\phi dx + \psi dy$

Q2. $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$

$\int_C \vec{F}\cdot\vec{dr}$; $C = x^2+y^2=a^2$ $\qquad \vec{dr} = dx\,\hat{i} + dy\,\hat{j}$

A2



$\vec{F}\cdot\vec{dr} = \sin y\,dx + x(1+\cos y)\,dy$

$\phi(x,y) = \sin y$
$\psi(x,y) = x(1+\cos y)$

$$\int_C (\phi\,dx + \psi\,dy) = \iint_R (\cancel{\cos}\ 1+\cos y - \cos y)\,dx\,dy$$

$$= \iint_R dx\,dy = \pi a^2$$

Stokes Theorem

Relates line integral with double integral.

$$\int_C \vec{F}\cdot\vec{dr} = \iint_S \operatorname{curl}(\vec{F})\cdot\hat{\eta}\,ds$$

$\hat{\eta}$ = unit normal external to the surface element s

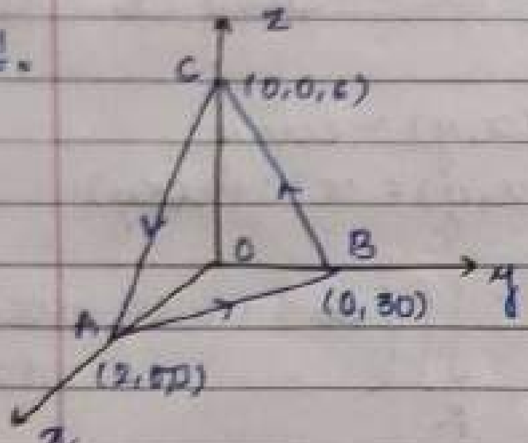
$$\hat{\eta} = \frac{\vec{\nabla}s}{|\vec{\nabla}s|}$$

Divergence Theorem

$$\iint_S \vec{F}\cdot ds = \iiint_V \operatorname{div}(F)\,dv \qquad dv = dx\,dy\,dz$$

Q. verify strokes theorem for
$\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$
over the surface of a Δ lamina w vertices
$(2,0,0)$; $(0,3,0)$; $(0,0,6)$

A.



$$LHS = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$$

$C = AB + BC + CA$

$$= \int_{AB} + \int_{BC} + \int_{CA} \left\{ (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz \right\}$$

C along AB $\Rightarrow z=0$; $\frac{x}{2} + \frac{y}{3} = 1$

$dz = 0$; $y = 3 - \frac{3}{2}x = \frac{6-3x}{2}$

$dy = -\frac{3}{2}\,dx$

$2 \le x \le 0$

$$\int_{x=2}^{0} \left[x + \frac{6-3x}{2}\right] dx + (2x-0)\left(-\frac{3}{2}\,dx\right) + 0$$

$$= \int_{x=2}^{0} \left(x + 3 - \frac{3x}{2}\right) dx + \left(-3x\,dx\right)$$

$$= \int_{x=2}^{0} \left(x + 3 - \frac{3x}{2} - 3x\right) dx$$

$$= \left[\frac{x^2}{2} + 3x - \frac{3x^2}{4} - \frac{3x^2}{2}\right]_2^0 = \left[-2 - \cancel{6} + 3 + \cancel{6}\right] = 1$$

along BC $\int_{BC} \vec{F}\cdot\vec{dr}$

$x=0 \Rightarrow dx=0$; $3 \le y \le 0$; $0 \le z \le 6$

Eqt of line BC $= \frac{y}{3} + \frac{z}{6} = 1$

$$y = \left(1 - \frac{z}{6}\right)3 = 3 - \frac{3z}{6}$$

$$= \frac{6-z}{2} \Rightarrow dy = \frac{-dz}{2}$$

$$\int_{BC} \vec{F}\cdot\vec{dr} = \int_{z=0}^{6} -z\left(\frac{-dz}{2}\right) + \left(\frac{6-z}{2} + z\right)dz \quad \text{—(ii)}$$

along CA :- $y=0 \Rightarrow dy=0$

$0 \le x \le 2$; $6 \le z \le 0$

$$\frac{x}{2} + \frac{z}{6} = 1 \Rightarrow z = \left(1 - \frac{x}{2}\right)6 = 6 - 3x$$

$$dz = -3dx$$

$$\int_{CA} \vec{F}\cdot\vec{dr} = \int_{x=0}^{x=2} x\,dx + (6-3x)(-3dx) = (-16)$$

(ii) $\int_{BC} \vec{F}\cdot\vec{dr} = \int_{z=0}^{6} \left(\frac{z}{2} + 3 - \frac{z}{2} + z\right)dz$

$= 36$

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RHS :- $\iint_S \text{curl}(\vec{F}) \hat{\eta}\, ds$

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$\hat{i}\left(\frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial z}(2x-z)\right) - \hat{j}\left(\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial z}(x+y)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(2x-z) - \frac{\partial}{\partial y}(x+y)\right)$$

$$= \hat{i}(2) + \hat{k}$$

$S : \frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \qquad \hat{\eta} = \frac{\vec{\nabla}S}{|\vec{\nabla}S|}$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{6}\right)$$

$$\vec{\nabla}S = \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{6}\hat{k}$$

$$\sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{36}} = \sqrt{\frac{9+4+1}{36}} = \sqrt{\frac{14}{36}} = \frac{\sqrt{14}}{6}$$

$$\hat{\eta} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$$

$$\iint \left(2\hat{i} + \hat{k}\right) \cdot \left(\frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}\right) \frac{ds}{\hat{\eta}_k} = \iint \left(\frac{7}{\sqrt{14}}\right) \frac{dx\,dy}{\hat{\eta}\cdot\hat{k}}$$

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} \qquad ds = \frac{dy\,dz}{\hat{n}\cdot\hat{i}} \qquad ds = \frac{dx\,dz}{\hat{n}\cdot\hat{j}}$$

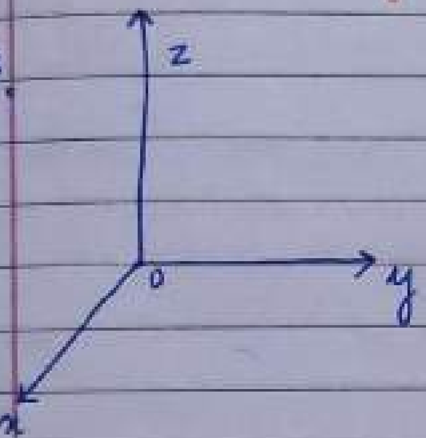
$$= \frac{7}{\cancel{\sqrt{14}}} \times \frac{1}{\cancel{2}} \times \cancel{2} \times 3$$

$$= \frac{7}{\cancel{\sqrt{14}}} \times 3 = 21.$$

Q. verify stokes theorem $\vec{F} = (x^2 + y - 4)\hat{i} + 3xy\hat{j} + (2xz + z^2)\hat{k}$

S :- Surface of hemisphere $x^2 + y^2 + z^2 = 16$ above xoy plane.

Ans.



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INSPECTION METHOD.

(1). $xdy + ydx = d(xy)$

(2). $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$

(3). $adx + bdy = d(ax+by)$

(4) $xdx = d\left(\dfrac{x^2}{2}\right)$